

# ON THE RANK OF ABELIAN VARIETIES

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## Abstract

## Full Text

UDC 511

MATHEMATICS

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# ON THE RANK OF ABELIAN VARIETIES

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Let  $\Gamma$  be an abelian variety over a number field  $k$ . Denote by  $r$  the rank of  $\Gamma$  over  $k$ , i.e. the number of generators of infinite order of the group of rational points of  $\Gamma$  over  $k$ . Our aim is to relate  $r$  to certain other invariants of  $\Gamma$ , under certain restrictive assumptions.

Fix a prime number  $p$ . Denote by  $A(L)$  the group of points of  $\Gamma$  rational over a field  $L$ , and by  $A_s$  the group of points of order  $p^s$  (in the algebraic closure of the field  $k$ ). Consider the set  $S$  of prime divisors of the field  $k$ , consisting of all divisors of  $p$  and of those prime divisors modulo which the variety  $\Gamma$  is degenerate. It is known that  $S$  is a finite set <sup>(1)</sup>. We impose on the field  $k$  the following condition: the field of definition of the points of order  $p^s$  is a  $p$ -extension of the field  $k$ . Denote by  $K$  the maximal  $p$ -extension of the field  $k$  in which only the divisors from  $S$  ramify. Denote the Galois group of  $K$  over  $k$  by  $F$ . For  $\pi \in S$ , denote by  $F_\pi$  the  $\pi$ -localization of the group  $F$  (i.e. the Galois group of the extension  $K_\pi$  of the local field  $k_\pi$  corresponding to the extension  $K$ , defined up to conjugacy).

**Proposition 1.** *There is the following exact sequence:*

$$0 \rightarrow \mathfrak{W} \rightarrow H^1(F, A(K)) \rightarrow \sum_{\pi \in S} H^1(F_\pi, A(K_\pi)),$$

where  $\mathfrak{W}$  is the  $p$ -component of the group of locally trivial principal homogeneous spaces over  $\Gamma$ .

The proof of this proposition can be carried over from <sup>(2)</sup>.

Next consider the standard exact sequence

$$0 \rightarrow B_s \rightarrow H^1(F, A_s) \xrightarrow{i_s} H^1(F, A(K)), \quad (1)$$

where the image of  $i_s$  consists precisely of the elements of the group  $H^1(F, A(K))$  of order  $p^s$ . The group  $B_s$  consists of the elements of the group  $H^1(F, A_s)$  of the form  $\varphi(\sigma) = P - P^\sigma$ , where  $p^s P$  is a rational point over  $k$ . Naturally,

$$B_s \approx A(k)/p^s A(k).$$

Denote

$$B_\infty = \varinjlim B_s; \quad A_\infty = \varinjlim A_s, \quad H^1(F, A_\infty) = \varinjlim H^1(F, A_s).$$

There is the following exact sequence:

$$0 \rightarrow B_\infty \rightarrow H^1(F, A_\infty) \xrightarrow{i} H^1(F, A(K)) \rightarrow 0.$$

In the local case one can also write an exact sequence analogous to sequence (1):

$$0 \rightarrow B_s^\pi \rightarrow H^1(F_\pi, A_s) \xrightarrow{i_s^\pi} H^1(F_\pi, A(K_\pi)),$$

but it can no longer be asserted that the image of  $i_s^\pi$  consists exactly of all elements of order  $p^s$ , since upon passing to the limit the map  $i^\pi$  need not be an epimorphism:

$$0 \rightarrow B_\infty^\pi \rightarrow H^1(F_\pi, A_\infty) \xrightarrow{i^\pi} H^1(F_\pi, A(K_\pi)).$$

Let us combine the exact sequences constructed into one diagram:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & \sqcup & & \\ & & & & \downarrow & & \\ 0 & \rightarrow & B_\infty & \rightarrow & H^1(F, A_\infty) & \xrightarrow{i} & H^1(F, A(K)) \rightarrow 0 \\ & & \downarrow & & \downarrow g & & \downarrow f \\ 0 & \rightarrow & \sum_{\pi \in S} B_\infty^\pi & \xrightarrow{j} & \sum_{\pi \in S} H^1(F_\pi, A_\infty) & \xrightarrow{\Sigma i^\pi} & \sum_{\pi \in S} H^1(F_\pi, A(K_\pi)). \end{array}$$

All the groups displayed are periodic  $p$ -groups having divisible groups as subgroups of finite index. By the dimension of a  $p$ -group we shall mean the rank of its divisible subgroup. Clearly,

$$\dim B_\infty = r, \quad \dim \sum_{\pi \in S} B_\infty^\pi = \sum_{\pi \in S} r_\pi,$$

where  $r_\pi$  is the dimension of the group of local points (not counting points of finite order) that are  $p$ -infinitely divisible in the field  $K_\pi$ . We denote the

dimension of  $\mathfrak{W}$  by  $\rho$ . Denote by  $\widetilde{B}$  the following group:  $\widetilde{B} = \text{Im } j \cap \text{Im } g$ , and by  $\tilde{r}$  the dimension of the group  $\widetilde{B}$ .

**Theorem 1.**  $r = \tilde{r} - \rho + \dim \text{Ker } g$ .

**Proof.** Since  $fi = \Sigma i^\pi \cdot g$ , we have

$$\tilde{r} = \dim \text{Im } g - \dim \text{Im } f.$$

On the other hand,

$$\dim \text{Im } f = \dim H^1(F, A(K)) - \rho$$

and

$$\dim H^1(F, A(K)) = \dim H^1(F, A_\infty) - r,$$

whence

$$\tilde{r} = \rho + r - \dim \text{Im } g - \dim H^1(F, A_\infty) = \rho + r - \dim \text{Ker } g,$$

as was required to prove.

To formulate a second, more special result we make two assumptions:

1. The kernel of  $g$  is finite.
2.  $p$  occurs in  $k$  as the  $n$ -th power of a unique prime divisor  $\pi$ , where  $n$  is the absolute degree of  $k$ , and  $S$  consists only of  $\pi$ . We shall also assume that the points  $A_1$  are defined over  $k$ .

The second assumption is, of course, satisfied only in very special cases. In the first part of the argument these conditions will not be used.

Denote by  $G_\pi$  the Galois group of the algebraic closure  $\bar{k}_\pi$  of the field  $k_\pi$ . Consider the exact sequence (see Proposition 1):

$$0 \rightarrow \mathfrak{W} \rightarrow H^1(F, A(K)) \xrightarrow{t} \sum_{\pi \in S} H^1(G_\pi, A(\bar{k}_\pi)) \rightarrow \mathfrak{B} \rightarrow 0,$$

where  $\mathfrak{B} = \text{Coker } t$ . Our task includes computing  $\mathfrak{B}$  under the indicated assumptions, which may be of interest in connection with the form of the result and its analogies with Cassels' results obtained for elliptic curves under the assumption that  $\rho = 0$  <sup>(3)</sup>.

Consider the group  $\hat{\mathfrak{B}}$ , the group of characters of the group  $\mathfrak{B}$ . It is a subgroup of the group  $\Sigma \widehat{A}(k_\pi)$ , as follows from the well-known Tate duality, and is the annihilator of the group  $\text{Im } t$ . We shall prove that the previously introduced group  $\widetilde{B}$  lies in the natural image of this annihilator in the group

$$\sum_{\pi \in S} H^1(F_\pi, A_\infty).$$

Indeed, the necessary scalar product can be computed as follows: take an element  $\varphi \in \widetilde{B}$ , find  $\Phi \in H^1(F, A_\infty)$  such that  $g\Phi = \varphi$ . Then take any element  $\Psi \in H^1(F, A_\infty)$ , compute  $z = \Phi \cup \Psi \in H^2(F, K)$ , and take the image of  $z$  in the group  $\Sigma H^2(F_\pi, K_\pi)$ . The result must be equal to zero. In fact, the restriction of the algebra  $z$  to any local field  $k$ , where  $q \notin S$ , is zero, since  $q$  does not ramify in  $K$ . Using the reciprocity law, we obtain that the sum of the invariants of this algebra over  $\pi \in S$  is equal to zero. Hence it follows that  $\dim \mathfrak{F} \geq r_s$ . This inequality does not depend on the assumptions formulated.

**Theorem 2.** Under assumptions 1 and 2, the equality  $r_s = \dim \mathfrak{F}$  holds.

This fact makes it possible to compute the group  $\widetilde{B}$ .

**Proof of Theorem 2.** Let us compute the dimension  $H^1(\widetilde{G}_\pi, A(\bar{k}_\pi))$ . It is equal to  $d(n+2)$ , where, we recall,  $n$  is the absolute degree of the field  $k$ , and  $d$  is the dimension of  $\Gamma$ . It is not hard to prove that

$$\dim H^1(F, A_\infty) \geq [\dim_{\mathbb{Z}_p} H^1(F, \mathbb{Z}_p) - \dim_{\mathbb{Z}_p} H^2(F, \mathbb{Z}_p)] \cdot 2d,$$

using the fact, noted in <sup>(2)</sup>, that the divisibility of the principal homogeneous space depends on the behavior of the elements of  $H^2(F, A_p)$ . The number of generators ( $\dim H^2(F, \mathbb{Z}_p)$ ) and an upper estimate for the number of relations (i.e., for  $\dim H^2(F, \mathbb{Z}_p)$ ) of the group  $F$  were found by I. R. Shafarevich <sup>(4)</sup>. Using his result, we obtain that

$$\dim H^1(F, A_\infty) \geq 2d(n - \tau) = d(2n - n + 2) = d(n + 2),$$

where  $\tau$  is the number of generators of the group of units of the field  $k$ , equal, by the assumptions on the field  $k$ , to  $n/2 - 1$ . To complete the proof of Theorem 2 it remains to note that

$$\begin{aligned} \dim B &= d(n + 2) - \dim H^1(F, A(K)) + \rho = \\ &= d(n + 2) - \dim \operatorname{Im} f = d(n + z) - \dim \operatorname{Im} g + r_s = \\ &= d(n + 2) - \dim H^1(F, A_\infty) + r_s \leq r_s. \end{aligned}$$

As an example of an application of the theorems proved, let us consider the curve  $y^l = x^k(1 - x)$ ,  $l \geq 2$  a prime number,  $k = 1, \dots, l - 2$ , over the field of division of the circle into  $l$  parts, and its Jacobian variety  $\Gamma$ . (The indicated curves are closely related to the Fermat curve <sup>(2)</sup>.) The set  $S$  consists of the single prime divisor  $\lambda$  of the number  $l$  in the field  $k$ . Theorem 1 gives  $r = r_s - \rho + \dim \operatorname{Ker} g$ . If  $l$  is regular, then it is easy to prove that  $\operatorname{Ker} g = 0$  and  $B_s = B_\infty^\lambda$ , and, consequently,  $r_s = r_\lambda$ —the dimension of the group of local points  $l$ -infinitely divisible in the field  $K_\lambda$ .

For the applicability of the second theorem, let us retain the assumption (for the irregular case) that  $\dim \operatorname{Ker} g = 0$ . The conditions formulated in the second

assumption are satisfied if one observes that  $\Gamma$  has “sufficiently many” complex multiplications and the kernel of one of them, the  $(l-1)$ -st power of which is multiplication by  $-l$ , is rational.

In conclusion, we note that the failure of even one of the  $r_\pi$  to coincide with the known maximal possible value entails, in the case when  $\Gamma$  is the Jacobian variety of some curve over the field of rational numbers, the finiteness of the number of rational points on the curve itself, which is not difficult to derive from the well-known theorem of Chabauty. This further increases the interest in the groups considered in the present paper.

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*Note: Figure translations are in progress. See original paper for figures.*

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