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Abstract

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MATHEMATICS

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ON THE QUASI-EQUIVALENCE OF BASES IN FINITE CENTERS OF HILBERT SCALES

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Let $\{x_k\}$ and $\{y_k\}$ be two bases in a linear topological space E . The basis $\{y_k\}$ is said to be quasi-equivalent to the basis $\{x_k\}$ if there exist a permutation $\{n_k\}$ of the natural series, a system of numbers σ_n , and an isomorphism K of the space E onto itself such that

$$x_k = \sigma_{n_k} K(y_{n_k}). \quad (1)$$

It is known ⁽¹⁾ that in a Hilbert space all unconditional bases are quasi-equivalent to one another (and in (1) one may dispense with the permutation). On the other hand, the quasi-equivalence of all bases* has been established for the so-called centers of nuclear Hilbert scales ^{(2)**} (in this case the permutation is already necessary). In the present paper the quasi-equivalence of unconditional bases in finite centers of completely continuous scales (not necessarily nuclear) is proved. Thus one of the problems posed in ⁽²⁾ is solved. The method proposed in the present paper not only makes it possible to dispense with the requirement that the space be nuclear, but also substantially simplifies the proof of quasi-equivalence (cf. ⁽²⁾, ⁽³⁾). The main theorem will be proved in Sec. 4°.

1°. Let E be a complete countably Hilbert space and let $\|x\|_p$, $p = 1, 2, \dots$, be a system of Hilbert norms defining the topology $\mathfrak{T}$ of the space E .

Under the indicated assumptions on the space E , the following holds.

Lemma 1. Let $\{x_k\}$ be an unconditional basis in E and let $\{x'_k\}$ be the biorthogonal system in E^* . Then the system of Hilbert norms

$$\|x\|_{0,p} = \left(\sum |x'_k(x)|^2 \|x_k\|_p^2 \right)^{1/2}$$

defines on E the original topology $\mathfrak{T}$.

This lemma is proved by applying Mazur's result ⁽⁴⁾ that unconditional convergence of the series $\sum x_k$ in a Hilbert space implies $\sum \|x_k\|^2 < \infty$, and by the usual arguments connected with the closed graph theorem.

2°. Using the concept of an n -diameter and its simplest properties ^(5,2), it is easy to prove the following lemma.

Lemma 2. Let $H_0 \subset H_1 \subset H_2 \subset H_3$ be a quadruple of Hilbert spaces with continuous embeddings; $\{e_k\}$ a common orthonormal basis in H_0 and H_3 ; $\{h_k\}$ a common orthonormal basis in H_1 and H_2 ; $\|e_k\|_{H_0} = \|h_k\|_{H_1} = 1$, $\|e_k\|_{H_3} = \mu_k \downarrow 0$, $\|h_k\|_{H_2} = \nu_k \downarrow 0$. Then there exists a constant $C < \infty$ such that:

$$\mu_k \leq C\nu_k.$$

The assumption on the existence of bases $\{e_k\}$ and $\{h_k\}$ with the properties indicated in the lemma is equivalent to the assumptions that the embedding operator from H_1 into H_2 is completely continuous and that H_0 is dense in H_3 .

* In a nuclear space all bases are absolute and, a fortiori, unconditional ⁽²⁾.

** The proof in ⁽²⁾ is carried out essentially by the method first applied in ⁽³⁾ in proving the quasi-equivalence of bases in the space A_r .

3°. Let $E_\alpha = E_\alpha(a_k) = \lim_{\lambda < \alpha} \text{pr } H_\lambda$ be a finite (i.e. $\alpha < \infty$) center of the Hilbert scale $H_\lambda = l_2(a_k^\lambda)$, $-\infty < \lambda < \infty$; $a_k \uparrow \infty$ —this condition ensures the complete continuity of the embedding H_λ into H_{λ_1} , $\lambda_1 < \lambda$, i.e. the space E_α will be Montel (6).*

Let $\{e_k\}$ be a system of unit vectors in $H_0 = l_2$; then

$$\|x\|_\lambda = \|x\|_{H_\lambda} = \left(\sum |(x, e_k)|^2 a_k^{2\lambda} \right)^{1/2},$$

where $(x, y) = (x, y)_{l_2}$. Obviously, $\{e_k\}$ is an unconditional basis in the space E_α ; we shall call it the principal basis in E_α . Finite centers possess the following important property, which will be used essentially below:

Lemma 3. Let the operator $T \in L(E_\alpha, H_{\lambda_0})^{**}$ for some $\lambda_0 < \alpha$, and at the same time $T \in L(H_\alpha, E_\alpha)$. Then $T \in L(E_\alpha, E_\alpha)$.

This assertion is a consequence of the interpolation theorem for operators in Hilbert scales (7).

4°. We shall prove the main theorem.

Theorem. Let the space E_α satisfy the assumptions of item 3° and let $\{x_k\}$ be an unconditional basis in E_α . Then $\{x_k\}$ is quasiequivalent to the principal basis $\{e_k\}$ in E_α .

Proof. For simplicity we put $\alpha = 0$. Let $\{x'_k\} \subset E_0^*$ be the system biorthogonal to the basis $\{x_k\}$. By Lemma 1, the system of norms

$$\|x\|_{0,\lambda} = \left(\sum |x'_k(x)|^2 \|x_k\|_\lambda^2 \right)^{1/2}, \quad \lambda < 0,$$

is equivalent to the original system of norms $\|x\|_\lambda$, $\lambda < 0$. Denote by $H_\lambda^{(0)}$ the completion of the space E_0 with respect to the norm $\|x\|_{0,\lambda}$. From the equivalence of the systems of norms it follows that there exists $\lambda_0 < 0$ such that, for $\lambda : \lambda_0 < \lambda < 0$, numbers $\mu_i = \mu_i(\lambda) < \infty$, $i = 1, 2$, can be found which ensure the following continuous embeddings:

$$H_{\mu_2} \subset H_\lambda^{(0)} \subset H_{\mu_1}. \quad (2)$$

Let now G_0 be some Hilbert space for which:

- a) the inclusions $H_0 \subset G_0 \subset E_0$ hold with continuous embeddings;
- b) $\{x_k\}$ is an orthogonal basis in G_0 (the existence of such a space G_0 will be shown later).

Normalizing the system $\{x_k\}$ in the space G_0 , we obtain the system $\{\sigma_k x_k\}$, $\sigma_k > 0$, $\|\sigma_k x_k\|_{G_0} = 1$. Fix some $\lambda : \lambda_0 < \lambda < 0$, and renumber the system $\{\sigma_k x_k\}$ so as to obtain a system $y_k = \sigma_{n_k} x_{n_k}$, ordered by decreasing norms in the space $H_\lambda^{(0)}$: $\|y_k\|_{0,\lambda} \downarrow 0$. It will be shown below that the linear operator T , defined by the relations

$$Tx = \sum_{k=1}^{\infty} (x, e_k)_0 e_k, \quad (3)$$

is an isomorphism of the space E_0 onto itself. The latter will mean the quasiequivalence of the bases $\{x_k\}$ and $\{e_k\}$, since $e_k = \sigma_{n_k} K(x_{n_k})$, $K = T^{-1}$.

First we establish the existence of a Hilbert space G_0 with the properties a) and b) indicated above. Let $\mathfrak{M}$ be the set of all nondecreasing functions $M(\lambda) > 0$, defined on $(-1, 0)$. For each function $M \in \mathfrak{M}$ consider the Hilbert space $G(M)$ of all ele-

* The scale H_λ itself in this case is naturally called completely continuous.

** The notation $T \in L(X, Y)$ means that either the operator T maps X linearly and continuously into Y , or (if it is not defined on all of X) it admits a closure to a linear continuous operator from X to Y .

ments from E having finite norm

$$\|x\|_{G(M)} = \left\{ \int_{-1}^0 \|x\|_{0,\lambda}^2 M(\lambda)^2 d\lambda \right\}^{1/2}.$$

It is not difficult to see that for every $M \in \mathfrak{M}$ the space $G(M)$ is continuously embedded in E_0 and the system $\{x_k\}$ is an orthogonal basis in $G(M)$. We shall show that one can choose $M = M_0 \in \mathfrak{M}$ so that H_0 is continuously embedded in $G(M_0)$. Indeed, the unit ball S of the space H_0 is a bounded set in E_0 ; therefore, for every $\lambda < 0$ there is $C(\lambda) < \infty$ such that

$$\|x\|_{0,\lambda} \leq C(\lambda) < \infty, \quad x \in S$$

(one may assume that $C(\lambda)$ is increasing). Then $S \subset U(M_0)$, where $M_0(\lambda) = C(\lambda)^{-1} \in \mathfrak{M}$. The latter means that H_0 is continuously embedded in $G(M_0)$. The space $G_0 = G(M_0)$ thus constructed is the desired one.

Finally, let us show that the operator T (3) is an isomorphism of E_0 onto itself. Since $\{e_k\}$ and $\{y_k\}$ are orthonormal bases, respectively, in H_0 and G_0 , the operator T is an isomorphism of the space H_0 onto G_0 . A fortiori,

$$T \in L(H_0, E_0), \quad T^{-1} \in L(H_0, E_0). \quad (4)$$

If we now establish that

$$T \in L(E_0, H_{\mu_1}), \quad T^{-1} \in L(E_0, H_{\mu_1}) \quad (5)$$

(for the definition of μ_1 , see (2)), then from (4) and (5), by Lemma 3, we obtain that T is an isomorphism of the space E_0 onto itself.

Thus it remains to prove the relations (5). Applying Lemma 2 to the quadruple of spaces $H_0 \subset G_0 \subset H_{\lambda}^{(0)} \subset H_{\mu_1}$ and to the bases $\{e_k\}$ and $\{y_k\}$, we obtain

$$\|e_k\|_{\mu_1} \leq C \|y_k\|_{0,\lambda}. \quad (6)$$

From the same lemma, applied to the quadruple of spaces $G_0 \subset H_{-\varepsilon} \subset H_{\mu_2} \subset H_{\lambda}^{(0)}$, $\varepsilon > 0$, $\mu_2 + \varepsilon < 0$, and to the bases $\{y_k\}$ and $\{\gamma_k e_k\}$, $\gamma_k = a_k^{\varepsilon}$, it follows that

$$\|y_k\|_{0,\lambda} \leq C_1 \|\gamma_k e_k\|_{\mu_2} = C_1 \|e_k\|_{\mu_2+\varepsilon}, \quad C_1 = C_1(\varepsilon) < \infty. \quad (7)$$

From (6) and (7) we obtain the inequalities

$$\begin{aligned} \|Tx\|_{0,\lambda} &= \left(\sum |(x, e_k)|^2 \|y_k\|_{0,\lambda}^2 \right)^{1/2} \leq C_1 \left(\sum |(x, e_k)|^2 \|e_k\|_{\mu_2+\varepsilon}^2 \right)^{1/2} = \\ &= C_1 \left(\sum |(x, e_k)|^2 a_k^{2(\mu_2+\varepsilon)} \right)^{1/2} = C_1 \|x\|_{\mu_2+\varepsilon}, \end{aligned}$$

$$\begin{aligned}\|T^{-1}y\|_{\mu_1} &= \left(\sum |(x, e_k)|^2 a_k^{2\mu_1}\right)^{1/2} = \left(\sum |(x, e_k)|^2 \|e_k\|_{\mu_1}^2\right)^{1/2} \leq \\ &\leq C \left(\sum |(x, e_k)|^2 \|y_k\|_{0,\lambda}^2\right)^{1/2} = C\|y\|_{0,\lambda}, \quad x = T^{-1}y.\end{aligned}$$

These inequalities mean, in view of (2), that $T \in L(H_{\mu_2+\varepsilon}, H_\lambda^{(0)}) \subset L(E_0, H_{\mu_1})$, $T^{-1} \in L(H_\lambda^{(0)}, H_{\mu_1}) \subset L(E_0, H_{\mu_1})$. The theorem is proved.

5°. Theorem 1 is equivalent to the following criterion for the isomorphism of the Köthe space $L(b_{n,p})$ ⁽²⁾ to the finite center of a completely continuous Hilbert scale $E_\alpha = E_\alpha(a_k)$.

Theorem 2. In order that the space $L(b_{n,p})$ be isomorphic to the space E_α , it is necessary and sufficient that there exist a permutation $\{n_k\}$ of the natural numbers and a sequence of numbers $\sigma_k > 0$ such that the spaces E_α and $L(C_{k,p})$, $C_{k,p} = \sigma_k a_{n_k,p}$, coincide set-theoretically.

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