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Abstract

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MATHEMATICS

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ON A GENERAL BOUNDARY-VALUE PROBLEM

FOR ONE SYSTEM OF EQUATIONS OF COMPOSITE TYPE

(Presented by Academician I. N. Vekua on 5 X 1967)

Let G be a simply connected domain in the plane $\zeta = \xi + i\eta$, bounded by a simple closed smooth curve γ , having no segments parallel to the axis $\xi = 0$, and possessing the property that every straight line $\xi = \text{const}$ passing through G intersects γ in exactly two points. Let N, M be the points of tangency of the curve γ with the straight lines $\xi = a, \xi = b$ ($a < b$). Taking as positive on γ the direction of motion that leaves the domain G on the left, denote by γ_1 the arc $\overline{MN}$. Consider in the domain G the system of equations

$$\begin{aligned} \partial\varphi/\partial\bar{\zeta} - \partial\varphi/\partial\zeta &= A_0(\zeta)\varphi + B_0(\zeta)\bar{\varphi} + C_0(\zeta)\psi + D_0(\zeta)\bar{\psi}, \\ \partial\psi/\partial\bar{\zeta} - q(\zeta)\partial\psi/\partial\zeta &= A_1(\zeta)\varphi + B_1(\zeta)\bar{\varphi} + C_1(\zeta)\psi + D_1(\zeta)\bar{\psi}; \\ \partial/\partial\bar{\zeta} &= \frac{1}{2}(\partial/\partial\xi + i\partial/\partial\eta), \quad \partial/\partial\zeta = \frac{1}{2}(\partial/\partial\xi - i\partial/\partial\eta) \end{aligned} \quad (\text{I})$$

with respect to the complex-valued unknown functions $\varphi(\zeta), \psi(\zeta)$, with given complex-valued coefficients. We shall assume that the coefficient $q(\zeta)$ of system (I) satisfies the condition $|q(\zeta)| \leq q_0 < 1$, is Hölder continuous together with its derivatives up to order n' inclusive in the domain $\bar{G} = G + \gamma$, is Hölder continuous together with its derivatives up to order $n' - 1$ and belongs to the class L_p , $p > 2$, in the whole ζ -plane, while the coefficients $A_j(\zeta), B_j(\zeta), C_j(\zeta), D_j(\zeta)$ will be regarded as Hölder continuous in $\bar{G}$ together with their derivatives up to order $n' - 1$, where n' is a certain positive integer, $n' = \max(m, n)$. System (I) is a system of composite type in the domain G with a double system of characteristics $\xi = \text{const}$. To a system of the form (I), in a neighborhood of each point of G , any system of the general form

$$\partial w_k/\partial\bar{\zeta} - q_k(\zeta)\partial w_k/\partial\zeta = \sum_{l=1}^2 \left(A_{kl}(\zeta)w_l(\zeta) + B_{kl}(\zeta)\overline{w_l(\zeta)} \right), \quad k = 1, 2,$$

with coefficients $q_k(\zeta)$ satisfying, in the indicated neighborhood, the conditions $|q_1(\zeta)| = 1$, $|q_2(\zeta)| \leq \text{const} < 1$, can be reduced.

Let m, n be positive integers. We pose the following problem:

Problem $D_{m,n}$. Find regular solutions of system (I) such that $\varphi(\zeta)$ is Hölder continuous up to order m inclusive, and $\psi(\zeta)$ is Hölder continuous up to order n inclusive in the closed domain $\bar{G}$, and satisfying the boundary conditions:

$$\text{Re} \left(\sum_{j=0}^m a_j^{(k)}(\zeta) \frac{\partial^j \varphi}{\partial \xi^j} + \sum_{j=0}^n b_j^{(k)}(\zeta) \frac{\partial^j \psi}{\partial \zeta^j} \right) = h_k(\zeta), \quad (D_{m,n})$$

where $\zeta \in \gamma$ for $k = 0$ and $\zeta \in \gamma_1$ for $k = 1, 2$, and $a_j^{(k)}(\zeta)$, $b_j^{(k)}(\zeta)$ are given complex-valued real functions respectively on γ for $k = 0$ and on γ_1 for $k = 1, 2$, Hölder continuous there. If pri-

take into account the required smoothness of the solutions of system (I) in the closed domain, then it is easy to see that the general boundary-value problem is reduced to the problem formulated:

$$\text{Re} \left(\sum_{0 \leq i+j \leq m} a_{ij}^{(k)}(\xi) \frac{\partial^{i+j} \varphi}{\partial \xi^i \partial \eta^j} + \sum_{0 \leq i+j \leq n} b_{ij}^{(k)}(\xi) \frac{\partial^{i+j} \psi}{\partial \xi^i \partial \eta^j} \right) = h_k(\xi).$$

Let $\bar{\gamma}_1$ be the closure of the arc γ_1 . The main results concerning the problem $D_{m,n}$ are contained in the following theorem:

Theorem. *If on the arc $\bar{\gamma}_1$ the inequality*

$$\Delta(\xi) = 2i \left(b_n^{(0)}(\xi) a_{12}(\xi) - b_n^{(1)}(\xi) a_{02}(\xi) + b_n^{(2)}(\xi) a_{01}(\xi) \right) \neq 0, \quad (\text{N})$$

$$a_{kl}(\xi) = \text{Im} \left(a_m^{(k)}(\xi) a_m^{(l)}(\xi) \right),$$

is satisfied, on the arc $\gamma - \gamma_1$ the inequality $b_n^{(0)}(\xi) \neq 0$ is satisfied, and at the points M, N the equalities $a_j^{(0)} = 0$, $2a_{12} = -1$, $j = 0, 1, \dots, m$, are satisfied, then the homogeneous problem $D_{m,n}^0$ ($h_k(\xi) \equiv 0$) can have only a finite number of linearly independent solutions, and for the solvability of the nonhomogeneous problem it is necessary and sufficient that a finite number of conditions be fulfilled; moreover, for the index of the problem $D_{m,n}$ the formula holds

$$\text{Ind}(D_{m,n}) = 2(\kappa + m + n) + 1,$$

where

$$\varkappa = \frac{1}{2\pi} \{\arg \Delta^*(\xi)\}_\gamma, \quad \Delta^*(\xi) = \begin{cases} i\Delta(\xi), & \xi \in \bar{\gamma}_1, \\ b_n^0(\xi), & \xi \in \gamma - \gamma_1, \end{cases}$$

in which $\{\arg \Delta^*(\xi)\}_\gamma$ denotes the increment of the function $\arg \Delta^*(\xi)$ when the contour γ is traversed once in the positive direction.

In the general case the proof of this theorem is cumbersome. Therefore we shall restrict ourselves to the case when system (I) is split with respect to the functions $\varphi(\xi)$, $\psi(\xi)$, i.e., when $C_0(\xi) = D_0(\xi) = A_1(\xi) = B_1(\xi) \equiv 0$. In this case $\varphi(\xi) = e^{-ip(\xi)}\varphi^0(\xi)$, where the function $\varphi^0(\xi)$ is a solution of the Volterra integral equation

$$\varphi^0(\xi) - \int_{\eta_0}^{\eta} K_0(\xi + i\sigma) \overline{\varphi^0(\xi + i\sigma)} d\sigma = \omega(\xi),$$

$$K_0(\xi) = -iB_0(\xi)e^{2i\operatorname{Re} p(\xi)}, \quad p(\xi) = \int_{\eta_0}^{\eta} A_0(\xi + i\sigma) d\sigma, \quad (1^\circ)$$

$\omega(\xi)$ is an arbitrary complex-valued function, Hölder-continuous together with its derivatives up to order m inclusive on the segment $[a, b]$, and η_0 is fixed so that the line $\eta = \eta_0$ either touches the domain G or intersects it. By the method of successive approximations, from (1°) we find

$$\varphi^0(\xi) = \Gamma_1(\xi)\omega(\xi) + \Gamma_2(\xi)\overline{\omega(\xi)}, \quad (2)$$

where the functions $\Gamma_j(\xi)$ satisfy the condition

$$|\Gamma_1(\xi)|^2 - |\Gamma_2(\xi)|^2 = 1. \quad (3)$$

Hence

$$\varphi(\xi) = \Gamma_1^0(\xi)\omega(\xi) + \Gamma_2^0(\xi)\overline{\omega(\xi)}, \quad \Gamma_j^0(\xi) = e^{-ip(\xi)}\Gamma_j(\xi). \quad (4)$$

Inequality (N) makes it possible to write the boundary conditions $(D_{m,n})$ in the form

$$\operatorname{Re} \left(i\Delta(\xi) \frac{\partial^n \psi}{\partial \xi^n} + \sum_{j=0}^{n-1} \Delta_j(\xi) \frac{\partial^j \psi}{\partial \xi^j} + \sum_{j=0}^{m-1} d_j(\xi) \frac{\partial^j \varphi}{\partial \xi^j} \right) = h^0(\xi), \quad \xi \in \gamma_1; \quad (5)$$

$$\frac{\partial^m \varphi}{\partial \xi^m} + \sum_{j=0}^{m-1} \left(a_j(\xi) \frac{\partial^j \varphi}{\partial \xi^j} + b_j(\xi) \frac{\partial^j \bar{\varphi}}{\partial \xi^j} \right) = \sum_{j=0}^{n-1} C_j(\xi) \frac{\partial^j \psi}{\partial \xi^j} + \sum_{j=0}^n e_j(\xi) \overline{\left(\frac{\partial^j \psi}{\partial \xi^j} \right)} + h^{(1)}(\xi), \quad \xi \in \gamma_1; \quad (6)$$

$$\operatorname{Re} \left(\sum_{j=0}^m a_j^{(0)}(\xi) \frac{\partial^j \varphi}{\partial \xi^j} + \sum_{j=0}^n b_j^{(0)}(\xi) \frac{\partial^j \psi}{\partial \xi^j} \right) = h_0(\xi), \quad \xi \in \gamma - \gamma_1 = \gamma_2, \quad (7)$$

where the functions $\Delta_j(\xi), d_j(\xi), a_j(\xi), b_j(\xi), c_j(\xi), e_j(\xi), h^0(\xi), h^{(1)}(\xi)$ are expressed in terms of the coefficients of condition $(D_{m,n})$ and can easily be written out.

Let $\eta = \sigma_j(\xi)$ be the equation of the curve γ_j in Cartesian coordinates. Then, taking into account equalities (3) and (4), from (6) for the function $\omega(\xi)$ we obtain an ordinary differential equation of order m :

$$\frac{d^m \omega}{d\xi^m} + \sum_{k=0}^{m-1} \left(A_k^0(\xi) \frac{d^k \omega}{d\xi^k} + B_k^0(\xi) \frac{d^k \bar{\omega}}{d\xi^k} \right) = f(\xi) \quad (a \leq \xi \leq b), \quad (8)$$

with right-hand side

$$f(\xi) = \sum_{k=0}^{n-1} C_k^0(\xi) \left(\frac{\partial^k \psi}{\partial \xi^k} \right)_{\zeta=\xi+i\sigma_1(\xi)} + \overline{D_k^0(\xi)} \left(\frac{\partial^k \bar{\psi}}{\partial \bar{\xi}^k} \right)_{\zeta=\xi+i\sigma_1(\xi)} + h^{(2)}(\xi), \quad (9)$$

where $A_k^0(\xi), B_k^0(\xi), C_k^0(\xi), D_k^0(\xi), h^{(2)}(\xi)$ are completely determined functions.

Let $\{\omega_j(\xi)\}_{j=1}^m$ be a fundamental system of solutions of the homogeneous equation

$$\frac{d^m \omega}{d\xi^m} + \sum_{k=0}^{m-1} A_k^0(\xi) \frac{d^k \omega}{d\xi^k} = 0,$$

$V(\xi)$ the Wronskian of the system $\{\omega_j(\xi)\}_{j=1}^m$, and $V_{ml}(\xi)$ the algebraic cofactor of the element in $V(\xi)$ lying at the intersection of the m -th row and the l -th column. Then it can be shown that differential equation (8) is equivalent to the following Volterra-type integral equation with respect to the function $\omega(\xi)$:

$$\omega(\xi) - \int_a^\xi k(\xi', \xi) \overline{\omega(\xi')} d\xi' = g(\xi) \quad (a \leq \xi \leq b), \quad (10)$$

where

$$k(\xi', \xi) = \sum_{k=0}^{m-1} (-1)^{k+1} \frac{\partial^k}{\partial \xi'^k} \sum_{l=1}^m V_{ml}(\xi') \omega_l(\xi) \frac{B_k^0(\xi')}{V(\xi')},$$

$$g(\xi) = \sum_{l=1}^m \omega_l(\xi) \int_a^\xi \frac{V_{ml}(\xi')}{V(\xi')} f(\xi') d\xi' + \sum_{l=1}^m \left(C_l + \sum_{k=1}^{m-1} k B_k^0(a) \overline{C_k} \right) \omega_l(\xi), \quad (11)$$

and C_l are complex constants: $C_l(d^{l-1}\omega/d\xi^{l-1})_{\xi=a}$.

Solving equation (10) by the method of successive approximations, we obtain for $\omega(\xi)$ the expression

$$\begin{aligned} \omega(\xi) = \sum_{j=1}^n \int_a^\xi \left(A_j^{(1)}(\xi', \xi) \left(\frac{\partial^j \psi}{\partial \xi^j} \right)_{\zeta=\xi'+i\sigma_1(\xi')} + B_j^{(1)}(\xi', \xi) \left(\frac{\overline{\partial^j \psi}}{\partial \bar{\xi}^j} \right)_{\zeta=\xi'+i\sigma_1(\xi')} \right) d\xi' + \\ + \sum_{j=1}^m (e_j^0(\xi) C_j + d_j^0(\xi) \overline{C_j}) + h(\xi), \end{aligned} \quad (12)$$

where the functions $A_j^{(1)}(\xi', \xi)$, $B_j^{(1)}(\xi', \xi)$, $e_j^0(\xi)$, $d_j^0(\xi)$, $h(\xi)$ are completely determined; moreover, $h(\xi) \equiv 0$ when $h^{(1)}(\xi) \equiv 0$.

Thus, using boundary condition (6), we have managed to express the function $\omega(\xi)$ in terms of the boundary values of the functions $\partial^j \psi / \partial \zeta^j$ and the constants $C_1, C_2, \dots, C_m$. But since $\psi(\zeta)$ is a regular solution of the elliptic equation $\partial \psi / \partial \bar{\zeta} - q(\zeta) \partial \psi / \partial \zeta = C_1(\zeta) \psi + D_1(\zeta) \bar{\psi}$, the representation (1) holds:

$$\psi(\zeta) = \psi_0(\zeta) + \iint_G (R_1(t, \zeta) \psi_0(t) + R_2(t, \zeta) \overline{\psi_0(t)}) dG_t, \quad (13)$$

where $\psi_0(\zeta)$ is the general solution of class $C_n(\overline{G})$ of the Beltrami equation $\partial \psi / \partial \bar{\zeta} - q(\zeta) \partial \psi / \partial \zeta = 0$. Making use now of the integral representa-

by I. N. Vekua (see (2), p. 275) for the function $\psi_0(\zeta)$:

$$\psi_0(\zeta) = \int_\gamma \mu(t) \left(1 - \frac{W(\zeta)}{W(t)} \right)^{n-1} \ln \left(1 - \frac{W(\zeta)}{W(t)} \right) ds + \int_\gamma \mu(t) ds + i\beta_0, \quad (14)$$

where $W = W(\zeta)$ is a homeomorphism of class C_ν^n of the Beltrami equation (1), $\mu(t)$ is an unknown real function, Hölder-continuous on γ , and β_0 is a

real constant; taking into account (4), (12), (13), we find that the boundary conditions (5), (7) are equivalent to the following singular integro-functional equation (cf. (4)):

$$K(\mu) \equiv \operatorname{Re} A^*(t_0) \cdot \mu(t_0) + \operatorname{Re} B^*(t_0) \cdot \mu[t(\alpha(s_0))] + \frac{1}{\pi} \operatorname{Im} A^*(t_0) \cdot \int_{\gamma} \frac{\mu(t) dt}{t - t_0} - \frac{1}{\pi} \operatorname{Im} B^*(t_0) \cdot \int_{\gamma} \frac{\mu[t(\alpha(s))] dt}{t - t_0} + T(\mu) = \sum_{j=0}^{2m} \gamma_j(t_0) \beta_j + h^*(t_0), \quad t_0 \in \gamma, \quad (15)$$

where $\alpha(s)$ is a real function of class C^1_ν establishing a homeomorphism between the parts $\gamma_1, \gamma - \gamma_1$ of the contour γ , determined from the equation $\xi[\alpha(s)] = \bar{\xi}(s)$; $T(\mu)$ is a completely continuous operator; β_j are real constants,

$$A^*(t) = \frac{(-1)^n (n-1)! \pi i a^*(t)}{(W(t))^{n-1} \theta(t)} \left(\frac{\partial W}{\partial t} \right)^{n-1}; \quad a^*(t) = \begin{cases} i\Delta(t), & t \in \gamma_1, \\ b_n^{(0)}(t), & t \in \gamma - \gamma_1; \end{cases}$$

$$B^*(t) = \frac{(-1)^n (n-1)! \pi i b^*(t)}{(W[t(\alpha(s))])^{n-1} \theta[t(\alpha(s))]} \left(\frac{\partial W}{\partial t} \right)^{n-1}_{t=t(\alpha(s))}; \quad b^*(t) = \begin{cases} 0, & t \in \gamma_1, \\ b_*(t), & t \in \gamma - \gamma_1, \end{cases}$$

$$b_*(t) = \left(a_m^{(0)}(t) \overline{\Gamma_1^0(t)} + a_m^{(0)}(t) \overline{\Gamma_2^0(t)} \right) \overline{\Gamma_1[t(\alpha(s))]} \overline{D_n[t(\alpha(s))]},$$

$$D_n(t) = -\frac{1}{\Delta(i)} \left(\overline{b_n^{(0)}(t)} b_{12}(t) - \overline{b_n^{(1)}(t)} b_{02}(t) + \overline{b_n^{(2)}(t)} b_{01}(t) \right),$$

$$b_{kl}(t) = \overline{a_m^{(k)}(t)} b_n^{(l)}(t) - \overline{b_n^{(k)}(t)} a_m^{(l)}(t).$$

Taking into account the inequalities (N) and $b_n^{(0)}(t) \neq 0$, $t \in \gamma - \gamma_1$, on the basis of the results on equations of the form (15) obtained in (3) (see also (4)), we are convinced of the validity of the theorem.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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