

ON FACTORIAL $\backslash(s\backslash)$ -MAPPINGS

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1968

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Abstract

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UDC 513.83

MATHEMATICS

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ON FACTORIAL s -MAPPINGS

(Presented by Academician P. S. Aleksandrov, 18 XII 1967)

The present note is devoted to the study of factorial s -mappings of spaces with a point-countable base onto spaces of point-countable type. All necessary definitions, as well as some comments, may be found in ⁽¹⁾.

Theorem 1*. *Let $f : X \rightarrow Y$ be a factorial s -mapping of a space X with a point-countable base onto a T_2 -space Y of point-countable type. Then the first axiom of countability holds in the space Y .*

We shall carry out the proof in several stages.

I. If Y' is a set closed in Y , and $X' = f^{-1}(Y')$, then $f|_{X'} : X' \rightarrow Y'$ is a factorial mapping; therefore the following lemma, proved by A. V. Arhangel'skii for completely regular spaces and by M. Choban for T_2 -spaces, allows us to conduct the proof under the assumption that Y is bicomact, which we shall do.

Lemma (see ⁽²⁾). *If a bicomactum B has countable character in a T_2 -space Y , and a point $y \in B \subseteq Y$ has countable character in the subspace B , then the point y has countable character also in the space Y .*

II. Let $m = \{y_k^i, i, k = 1, 2, \dots\} \subseteq Y$ and let

$$y \in [\{y_k^i, i = 1, 2, \dots, k > l\}] \setminus \{y_k^i, i = 1, 2, \dots, k > l\}$$

for every l . Then from the set m one can choose a sequence of points

$$\{y_{k(n)}^{i(n)}, n = 1, 2, \dots, k(n+1) > k(n)\}$$

such that

$$[\{y_{k(n)}^{i(n)}\}] = \{y_{k(n)}^{i(n)}\} \cup \{y\}.$$

As the sum of a countable number of separable sets, the set $f^{-1}(m)$ is separable, and therefore, for any point-countable base Δ of the space X , the set Δ^* of those of its elements which intersect $f^{-1}(m)$ is at most countable. Let

$$g = \{Y \setminus [f(\delta)], \delta \in \Delta^*, y \in [f(\delta)]\}.$$

By construction, g is a countable family of open sets in Y containing the point y . We shall suppose the elements of the family g enumerated, i.e. let $g = \{G_n\}$. We construct by induction a family $\gamma = \{\Gamma_n\}$. As Γ_1 take G_1 . Suppose open sets $\Gamma_1, \dots, \Gamma_{n-1}$ have been constructed. As Γ_n take an arbitrary open set satisfying the relation

$$y \in \Gamma_n \subseteq [\Gamma_n] \subseteq \Gamma_{n-1} \cap G_n.$$

Let

$$B = \bigcap_{n=1}^{\infty} \Gamma_n.$$

Since $[\Gamma_n] \subseteq \Gamma_{n-1}$, it follows that

$$B = \bigcap_{n=1}^{\infty} [\Gamma_n],$$

and consequently B is a bicom pactum. The family $\gamma = \{\Gamma_n\}$ is a certain base of neighborhoods** of the bicom pactum B : for any open set U containing B , beginning with some number N ,

$$[\Gamma_n] \subseteq U,$$

for otherwise $[\Gamma_n] \setminus U$ would be a decreasing sequence of nonempty bicom pacta; their intersection would be nonempty and would lie, on the one hand,

* *Note added in proof.* Very recently the author obtained a considerably stronger result.

Theorem 1'. *A T_2 -space of point-countable type which is a factorial s -image of a space with a point-countable base itself has a point-countable base.*

Theorem 1' completely covers almost all results known to the author in this direction, and also gives an answer to some of the questions that had remained open.

** In other words, for any open set U containing the set B , there is an element Γ of the family γ such that $B \subseteq \Gamma \subseteq U$.

in $\bigcap_{n=1}^{\infty} [\Gamma_n] = B$, and on the other—in $Y \setminus U$, which contradicts the inclusion $B \subseteq U$.

We construct by induction the sequence $\{y_{k(n)}^i, n = 1, 2, \dots\}$. As $y_{k(1)}^i$ take any point of the set $\{y_k^i\}$. Suppose the points $y_{k(1)}^i, \dots, y_{k(n-1)}^i$ have been chosen. Since, by assumption, $y \in [\{y_k^i; k > k(n-1)\}]$, it follows that $\Gamma_n \cap \{y_k^i, k > k(n-1)\} \neq \emptyset$. As $y_{k(n)}^i$ take any point of the set $\Gamma_n \cap \{y_k^i, k > k(n-1)\}$. Since every neighborhood of the set B contains all the points $y_{k(n)}^i$, beginning with some one of them, there are in the bicom pactum B (and only there—since a closed set and a point in a bicom pactum are separable) points that are limit points for this sequence. We shall show that y is the unique limit point. For this it suffices to show that the set

$$f^{-1}(\{y_{k(n)}^i\} \cup \{y\}) = Q$$

is closed. Suppose this is not so. Then there exists a point $x' \in [Q] \setminus Q$. Let $y' = f(x') \neq y$. Since Y , by assumption (see I), is a bicom pactum, there exists an open set $V \ni y'$ such that $y \in [V]$. Take an element δ of the base Δ with $x' \in \delta \subseteq f^{-1}(V)$. Since $x' \in [Q] \subseteq [f^{-1}(m)]$, we have $\delta \in \Delta^*$. Moreover, $f(\delta) \subseteq [f(\delta)] \subseteq [V]$ and $y' \in [V]$; therefore $Y \setminus [f(\delta)]$ is an element of the family g that does not contain the point y' . Hence $y' \in B$. But, as was already noted, the sequence $\{y_{k(n)}^i\}$ has no limit points outside B . Thus the supposition that there exists a point $x' \in [Q] \setminus Q$ has led us to a contradiction and, consequently, y is indeed the unique limit point of the sequence $\{y_{k(n)}^i\}$, as required.

III. We shall show that the mapping f is pseudo-open.

Suppose this is not so. Then there exists an open set $V \subseteq X$ such that $\text{Int } f(V)$ does not contain all points whose preimages lie entirely in V . Consider $U = \text{Int } f^{-1}f(V)$. Since, obviously, $V \subseteq U$ and $f(V) = f(U)$, the set $\text{Int } f(U) = \text{Int } f(V)$ also does not contain all points whose preimages lie entirely in U . It follows that there exist points

$$\begin{aligned} y_k^i &\in Y \quad (i, k = 1, 2, \dots), & f^{-1}(y_k^i) &\subseteq X \setminus U, \\ y_k &\in Y, \\ x_k^i &\in f^{-1}(y_k^i), & x_k &\in f^{-1}(y_k), \\ x_k^i &\rightarrow x_k \quad * \text{ as } i \rightarrow \infty, \\ x_k' &\in f^{-1}(y_k), \\ y &\in Y, & f^{-1}(y) &\subseteq U, \\ x_k' &\rightarrow x \in f^{-1}(y). \end{aligned}$$

Indeed, otherwise we would have that $f^{-1}f(X \setminus U)$ is a closed set, as is

$$f(f^{-1}f(X \setminus U)) = f(X \setminus U),$$

which contains not a single point whose preimage lies entirely in U . Then all such points would lie in the open set

$$Y \setminus f(X \setminus U) \subseteq f(U) = f(V),$$

which contradicts our supposition.

By II, from the set $\{y_k^i\}$ we can choose a sequence $\{y_{k(n)}^i, n = 1, 2, \dots\}$ such that

$$[\{y_{k(n)}^i\}] = \{y_{k(n)}^i\} \cup \{y\}.$$

It follows that the set $f^{-1}(\{y_{k(n)}^i\})$ has no limit points not belonging to it and lying outside $f^{-1}(y)$. But since $f^{-1}(\{y_{k(n)}^i\}) \subset X \setminus U$, where the set $X \setminus U$ is closed and

$$f^{-1}(y) \cap (X \setminus U) = \emptyset,$$

the set $f^{-1}(\{y_{k(n)}^i\})$ has no limit points in $f^{-1}(y)$ either. But then the set $\{y_{k(n)}^i\}$ is closed, which contradicts the inclusion $y \in [\{y_{k(n)}^i\}]$. The contradiction obtained proves that the mapping f is pseudo-open.

IV. Let $y \in Y$, and let $\{H_n\}$ be an arbitrary countable covering of the set $f^{-1}(y)$ by open subsets of X . We shall show that from it one can choose

* $x_i \rightarrow x$ if every neighborhood of the point x contains all the points x_i , beginning with some one of them.

a finite number of elements $\{H_1, \dots, H_n\}$ such that

$$y \in \text{Int } f(H_1 \cup \dots \cup H_n).$$

Suppose that this is not so; then, since the mapping f is pseudo-open, for any k there exist sequences of points

$$y_k^i \in Y \quad (i = 1, 2, \dots), \quad f^{-1}(y_k^i) \cap (H_1 \cup \dots \cup H_k) \neq \emptyset,$$

$$x_k^i \in f^{-1}(y_k^i), \quad x_k^i \rightarrow x_k \in f^{-1}(y).$$

In accordance with II, choose from the set $\{y_k^i\}$ a sequence $\{y_{k(n)}^{i(n)}\}$ such that

$$[\{y_{k(n)}^{i(n)}\}] = \{y_{k(n)}^{i(n)}\} \cup \{y\}.$$

Take an arbitrary $x \in f^{-1}(y)$; then, for some N , $x \in H_N$, and H_N is a neighborhood of the point x which does not meet the sets $f^{-1}(y_{k(n)}^{i(n)})$ as soon as $k(n) \geq N$, and consequently

$$x \in [f^{-1}\{y_{k(n)}^{i(n)}\}].$$

By II the set $f^{-1}(\{y_{k(n)}^{i(n)}\})$ has no limit points outside $f^{-1}(y)$, and therefore the set $f^{-1}(\{y_{k(n)}^{i(n)}\})$ is closed. Consequently, the set

$$\{y_{k(n)}^{i(n)}\} = ff^{-1}(\{y_{k(n)}^{i(n)}\})$$

is closed as well, which contradicts the equality

$$[\{y_{k(n)}^{i(n)}\}] = \{y_{k(n)}^{i(n)}\} \cup \{y\}.$$

The assertion is proved.

V. The set Δ' of elements of the base Δ which meet $f^{-1}(y)$ is at most countable. Therefore the family of all finite subsets of the set Δ' is also at most countable, as is the family

$$\beta = \{\text{Int } f(\delta_1 \cup \dots \cup \delta_n), \delta_1, \dots, \delta_n \in \Delta', n = 1, 2, \dots\}.$$

If we show that this family is a certain base at the point, then theorem 1 will thereby be proved. Let V be an open subset of Y containing the point y . For each point $x \in f^{-1}(y)$ there is an element δ of the base Δ such that

$$x \in \delta \subseteq f^{-1}(V).$$

The set of all such elements is at most countable; they all belong to Δ' and together cover the set $f^{-1}(y)$. By IV, from this family one can choose a finite number of elements $\delta_1, \dots, \delta_n$ such that

$$y \in \text{Int } f(\delta_1 \cup \dots \cup \delta_n),$$

i.e. $\text{Int } f(\delta_1 \cup \dots \cup \delta_n)$ is an element of the family β containing the point y and contained in the set V , as was required.

From the corollaries of the theorem just proved we give one (see (2)).

Theorem 2. Let $f : X \rightarrow Y$ be a factor s -mapping of a metric space X onto a regular separable space Y of point-countable type. Then the space Y is metrizable.

In particular:

Corollary. A separable bicomactum which is a factor s -image of a metric space is metrizable.

In conclusion I express my gratitude to my scientific adviser A. V. Arhangel'skii, and also to M. Choban for valuable discussions.

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Received
15 XII 1967

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1. A. V. Arhangel'skii, *UMN*, **21**, no. 4, 133 (1966).
2. M. Choban, *DAN*, **166**, no. 3, 562 (1966).

Note: Figure translations are in progress. See original paper for figures.

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