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Abstract

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MATHEMATICS

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THEOREMS ON HOMEOMORPHISMS IMPLEMENTED BY ELLIPTIC OPERATORS

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Theorems on homeomorphisms for elliptic operators were established in various spaces in the works ⁽¹⁻⁸⁾ (see also ⁽⁹⁾, pp. 170–177, 245–265, and the survey ⁽¹⁰⁾); related questions were also studied in ⁽¹¹⁾. In the present paper a number of new theorems on homeomorphisms is established, and the relation between various theorems on homeomorphisms is studied. It turns out that many known theorems, and a number of new theorems, on homeomorphisms can essentially be obtained from the theorem of ⁽⁵⁾ by means of a corresponding “gluing” of elements in the space of preimages and the space of images. We note that in ⁽⁸⁾ the idea was expressed of the possibility of applying the gluing method to obtain new theorems on homeomorphisms.

1. Let G be a bounded domain of the space E_n , Γ its boundary, $\bar{G} = G \cup \Gamma$. In $\bar{G}$ let there be given a properly elliptic differential expression L of order $2m$ with complex coefficients, and on Γ a system of m differential expressions $\{B_j\}_{j=1}^m$ of orders m_j ($m_j \leq 2m - 1$), which we assume to be normal in the sense of Aronszajn–Milgram–Schechter (see ^(12,2,9)) and covering L . For simplicity we assume the coefficients of all differential expressions considered in the paper and the boundary Γ to be infinitely smooth. We introduce the necessary function spaces. Let $l \geq 0$ be an integer, $1 < p < \infty$, $p' = p/(p - 1)$; $W_p^l(G)$ is the Sobolev space; $W_{p'}^{-l}(G)$ is the space conjugate to $W_p^l(G)$ with respect to $(u, v) = \int_G u \bar{v} dx$ ^(11,5);

$\|u\|_{s,p}$ is the norm in $W_p^s(G)$ (s an integer).

If $l \geq 1$, then $W_p^{l-1/p}(\Gamma)$ denotes the completion of the set $C^\infty(\Gamma)$ in the norm $\langle \varphi \rangle_{l-1/p,p} = \inf \|u\|_{l,p}$, where the infimum is taken over all $u \in W_p^l(G)$ equal to φ on Γ ; $W_{p'}^{-(l-1/p)}(\Gamma)$ is the space conjugate to $W_p^{l-1/p}(\Gamma)$ with respect to $\langle u, v \rangle = \int_\Gamma u \bar{v} dx$ ^(12,5). If $s_0, s_1, \dots, s_t$ are arbitrary integers, then we denote

$$K_{(s_0, s_{j-1}/p, p)}^{t+1} = W_p^{s_0}(G) + \sum_{j=1}^t W_p^{s_{j-1}/p}(\Gamma)$$

(here the upper index $t + 1$ is equal to the number of summands in the direct sum). If

$a = (a_0, a_1, \dots, a_m) \in K_{(-s_0, -(s_{j-1}/p), p')}^{m+1}$, and
 $v = (v_0, v_1, \dots, v_m) \in K_{(s_0, s_{j-1}/p, p)}^{m+1}$, then put

$$[a, v] = (a_0, v_0) + \sum_{j=1}^m \langle a_j, v_j \rangle.$$

Since the system of boundary expressions $\{B_j\}_{j=1}^m$ is, by assumption, normal, it can be supplemented by expressions $\{C_j\}_{j=1}^m$ to a Dirichlet system of order $2m$. But then there exists a “conjugate” system of boundary expressions $\{B'_j, C'_j\}_{j=1}^m$, also forming a Dirichlet system of order $2m$, such that Green’s formula holds (see (12,2,9))

$$(Lu, v) + \sum_{j=1}^m \langle B_j u, C'_j v \rangle = (u, L^+ v) + \sum_{j=1}^m \langle C_j u, B'_j v \rangle \quad (u, v \in C^{2m}(\bar{G})). \quad (1)$$

Here the orders m_j, l_j, m'_j, l'_j of the expressions B_j, C_j, B'_j, C'_j satisfy the equality $m_j + l'_j = m'_j + l_j = 2m - 1$. For arbitrary $s \geq 0$ and $1 < p < \infty$ we define the operators $\mathcal{L}_{s,p}, \mathcal{L}_{s,p}^+ : \mathcal{D}(\mathcal{L}_{s,p}) = \mathcal{D}(\mathcal{L}_{s,p}^+) = W_p^{2m+s}(G)$;

$$\mathcal{L}_{s,p} u = (Lu, B_1 u, \dots, B_m u) \in K_{(s, 2m+s-m_j-1/p, p)} \equiv K_{s,p}(G);$$

$$\mathcal{L}_{s,p}^+ v = (L^+ v, B'_1 v, \dots, B'_m v) \in K_{(s, 2m+s-m'_j-1/p, p)}^{m+1} \equiv K'_{s,p}(G).$$

In this paper, for simplicity, we shall assume that the defect is absent*. Then the operators $\mathcal{L}_{s,p}$ and $\mathcal{L}_{s,p}^+$ map $W_p^{2m+s}(G)$ homeomorphically onto $K_{s,p}(G)$ and $K'_{s,p}(G)$, respectively.

Let now s be an arbitrary integer. By $\widehat{W}_p^s(G)$ (^{4, 5, 8, 9}) we denote the completion of the set $C^\infty(\bar{G})$ in the norm

$$|||u|||_{s,p} = \left(\|u\|_{s,p}^p + \sum_{j=1}^{2m} \left\langle \frac{\partial^{j-1} u}{\partial \nu^{j-1}} \right\rangle_{s-j+1-1/p, p}^p \right)^{1/p} \quad (\nu \text{ is the normal to } \Gamma). \quad (2)$$

It is clear that if $s \geq 2m$, then the norms $\| \cdot \|_{s,p}$ and $\| \cdot \|_{s,p}$ are equivalent; for $s < 2m$ they are not equivalent. The closure S of the mapping

$$u \rightarrow (u|_G, u|_\Gamma, \dots, \partial^{2m-1}u/\partial\nu^{2m-1}|_\Gamma) \quad (u \in C^\infty(\bar{G})),$$

considered as acting from $\widehat{W}_p^s(G)$ into $K_{(s, s-j+1-1/p, p)}^{2m+1}$ ($j = 1, \dots, 2m$; s arbitrary integer), maps isometrically $\widehat{W}_p^s(G)$ onto a subspace of the direct sum $K_{(s, s-j+1-1/p, p)}^{2m+1}$ (for $s \leq 0$, $S\widehat{W}_p^s(G) = K_{(s, s-j+1-1/p, p)}^{2m+1}$). The components of the vector Su will also be called the components of the element $u \in \widehat{W}_p^s(G)$. For every differential expression M of order $t \leq 2m$ defined in $\bar{G}$, the operator $u \rightarrow Mu$ ($u \in C^\infty(\bar{G})$) acts continuously from $\widehat{W}_p^s(G)$ to $W_p^{s-t}(G)$; similarly, for any boundary differential expression B of order $t \leq 2m-1$, the operator $u \rightarrow Bu|_\Gamma$ ($u \in C^\infty(\bar{G})$) acts continuously from $\widehat{W}_p^s(G)$ to $W_p^{s-t-1/p}(\Gamma)$ (s is any integer) (^{4, 5, 8, 9}). If $u_0 \in \widehat{W}_p^s(G)$, then by Mu_0 (Bu_0) we denote the value, on the element u_0 , of the closure of the mapping $u \rightarrow Mu$ ($u \rightarrow Bu|_\Gamma$) ($u \in C^\infty(\bar{G})$), considered as acting from $\widehat{W}_p^s(G)$ to $W_p^{s-t}(G)$ ($W_p^{s-t-1/p}(\Gamma)$). In accordance with this convention, for arbitrary $u \in \widehat{W}_p^s(G)$ we still denote

$$Su = (u|_G, u|_\Gamma, \dots, \partial^{2m-1}u/\partial\nu^{2m-1}|_\Gamma).$$

We also note that the norm (2) is equivalent to the norm (⁸)

$$\{u\}_{s,p} = \left(\|u\|_{s,p}^p + \sum_{j=1}^m \langle B_j u \rangle_{s-m_j-1/p, p}^p + \sum_{j=1}^m \langle C_j u \rangle_{s-l_j-1/p, p}^p \right)^{1/p}. \quad (3)$$

2. In the present paper the following is used essentially.

Theorem 1 (⁵) (see also (^{4, 8, 9})). *For every integer s , the closure $\mathcal{L}_{s,p}$ of the mapping*

$$u \rightarrow (Lu, B_1 u, \dots, B_m u) \quad (u \in C^\infty(\bar{G})),$$

considered as acting from $\widehat{W}_p^{2m+s}(G)$ to $K_{s,p}(G)$, establishes a homeomorphism between these spaces.

The use of Theorem 1 for obtaining other theorems on homeomorphisms is based on the following simple lemmas.

Lemma 1. *Let B_1 and B_2 be Banach spaces, and let T be a linear operator mapping B_1 homeomorphically onto B_2 ; let E_1 be a subspace of B_1 , and $E_2 = TE_1$. Then the operator T naturally defines a linear operator T' mapping the quotient space B_1/E_1 homeomorphically onto B_2/E_2 .*

If Q_2 is a Banach space, $Q_2 \subset B_2$ (topological embedding), then $Q_1 = T^{-1}Q_2$ will be a linear (generally speaking, nonclosed) subset of B_1 . Introduce in Q_1 the graph norm

$$\|x\|_{Q_1}^T = \|x\|_{B_1} + \|Tx\|_{Q_2}$$

* All the results of the paper are also valid in the presence of a defect. In this case, it is only necessary in the theorems to replace the spaces of images and preimages by their corresponding subspaces.

($x \in Q_1$). With respect to this norm Q_1 becomes a Banach space, which we denote by Q_1^T . The restriction of the operator T to Q_1 will also be denoted by T .

Lemma 2. *The operator T maps Q_1^T homeomorphically onto Q_2 . If R_2 is a linear submanifold of Q_2 , dense in Q_2 , then $R_1 = T^{-1}R_2$ is dense in Q_1^T . In this case the closure of the mapping $x \rightarrow Tx$ ($x \in R_1$), regarded as acting from Q_1^T to Q_2 , establishes a homeomorphism $Q_1^T \rightarrow Q_2$.*

We shall first apply Lemma 1, assuming that $B_1 = \widehat{W}_p^{2m+s}(G)$, $B_2 = K_{s,p}(G)$ (s an integer), and T is the operator $\mathfrak{L}_{s,p}$ occurring in Theorem 1. Choosing the subspace E_1 in various ways and putting $E_2 = TE_1$, we obtain various theorems on homeomorphisms.

3. Let $E_1 = E_{2m+s,p}^1$ be the subspace of $\widehat{W}_p^{2m+s}(G)$ consisting of elements $u \in \widehat{W}_p^{2m+s}(G)$ whose first component is equal to zero. It is clear that $\widehat{W}_p^{2m+s}(G)/E_1 = W_p^{2m+s}(G)$. From Green's formula (1) it follows easily that $E_2 = E_{s,p}^2 = \mathfrak{L}_{s,p}E_1$ consists of those and only those $F \in K_{s,p}(G)$ for which

$$[F, V] = 0 \quad (V \in \mathfrak{M}' = \{(v, C'_1 v, \dots, C'_m v) : v \in C^\infty(\overline{G}), B'_j v|_\Gamma = 0 \\ (j = 1, \dots, m)\}). \quad (4)$$

Thus the following theorem is valid.

Theorem 2. *For each integer s , the closure $\mathfrak{L}_{s,p}$ of the mapping*

$$u \rightarrow (Lu, B_1 u, \dots, B_m u) \quad (u \in C^\infty(\overline{G})),$$

regarded as acting from $\widehat{W}_p^{2m+s}(G)/E_1 = W_p^{2m+s}(G)$ to $K_{s,p}(G)/E_2$, establishes a homeomorphism between these spaces.

Let us note that $K_{s,p}(G)/E_2$ is the space adjoint with respect to $[\cdot, \cdot]$ to the closure in $K_{(-s, -s-l'_j-1/p', p')}^{m+1}$ of the set $\mathfrak{M}'$ (see ⁽¹³⁾, Ch. 4, §5, item 4).

4. Let now $E_1 = \widetilde{E}_{2m+s,p}^1$ be the subspace of $\widehat{W}_p^{2m+s}(G)$ consisting of elements $u \in \widehat{W}_p^{2m+s}(G)$ for which $u|_G = 0$, $B_{ju}|_\Gamma = 0$ ($j = 1, \dots, m$). From the equivalence of the norms (2), (3) it follows that $\widehat{W}_p^{2m+s}(G)/E_1 = T_p^{2m+s}(G)$ coincides with the completion of the set $C^\infty(\overline{G})$ in the norm

$$\left(\|u\|_{2m+s,p}^p + \sum_{j=1}^m \langle B_{ju} \rangle_{2m+s-m_j-1/p,p}^p \right)^{1/p}. \quad (5)$$

From Green's formula (1) it follows easily that $F \in K_{s,p}(G)$ belongs to $E_2 = \widetilde{E}_{s,p}^2 = \mathfrak{L}_{s,p}E_1$ if and only if $F = (f, 0, \dots, 0)$,

$$(f, v) = 0 \quad (v \in C^\infty(\text{pr})^+ = \{v : v \in C^\infty(\overline{G}); B'_j v|_\Gamma = 0 \ (j = 1, \dots, m)\}),$$

therefore

$$K_{s,p}(G)/E_2 = W_p^s(G)/M \dot{+} \sum_{j=1}^m W_p^{2m+s-m_j-1/p}(\Gamma) = W'_{s,p}(\text{pr})^+ \dot{+} \sum_{j=1}^m W_p^{2m+s-m_j-1/p}(\Gamma),$$

where M is the subspace of $W_p^s(G)$ consisting of elements $f \in W_p^s(G)$ such that $(f, v) = 0$ ($v \in C^\infty(\text{pr})^+$), while $W'_{s,p}(\text{pr})^+$ is the space adjoint with respect to $(\cdot, \cdot)$ to the closure in $W_p^{-s}(G)$ of the set $C^\infty(\text{pr})^+$. Thus the following is valid.

Theorem 3. *For each integer s , the closure $T_{s,p}$ of the mapping*

$$u \rightarrow (Lu, B_1 u, \dots, B_m u) \quad (u \in C^\infty(\overline{G})),$$

regarded as acting from $T_p^{2m+s}(G)$ to

$$W'_{s,p}(\text{pr})^+ \dot{+} \sum_{j=1}^m W_p^{2m+s-m_j-1/p}(\Gamma),$$

establishes a homeomorphism between these spaces.

Let us note that if $2m+s-m_j > 0$ ($j = 1, \dots, m$), then the norm (5) is equivalent to the norm $\|u\|_{2m+s,p}$ and $T_p^{2m+s}(G) = W_p^{2m+s}(G)$; therefore Theorem 3 is a generalization of the Lions-Magenes theorem (see ⁽¹⁰⁾, theo-

theorem 6.22), established for the special case of Dirichlet boundary conditions. From Theorem 3 there also follows directly the theorem on homeomorphisms of the work [3]. With the aid of Lemma 1 one can also obtain from Theorem 1 the theorem on homeomorphisms of the work [8].

5. We shall now use Lemma 2 and Theorem 3 in order to obtain, for the case of integral s , the Lions-Magenes theorem on homeomorphisms (see [2], Theorem 5.4). Let in Lemma 2

$$B_1 = T_p^{2m+s}(G), \quad B_2 = W'_{s,p}(\Gamma)^+ + \sum_{j=1}^m W_p^{2m+s-m_j-1/p}(\Gamma)$$

($s < 0$ an integer), and let T be the operator $T_{s,p}$ occurring in Theorem 3. Let

$$Q_2 = L_p(G) + \sum_{j=1}^m W_p^{2m+s-m_j-1/p}(\Gamma) = K_{(0, 2m+s-m_j-1/p, p)},$$

$$Q_1 = T_{s,p}^{-1} Q_2.$$

Introduce in Q_1 the graph norm

$$\|u\|_{Q_1^T} = \|u\|_{2m+s,p} + \|Lu\|_{0,p} + \sum_{j=1}^m \langle B_j u \rangle_{2m+s-m_j-1/p}. \quad (6)$$

The completion of $C^\infty(\overline{G})$ with respect to the norm (6) will be denoted by $W_{L,\{B_j\}}^{2m+s,p}(G)$. From Lemma 2 it follows directly that

Theorem 4. *For every integer $s < 0$, the closure of the mapping $T_{s,p}$ of $u \mapsto (Lu, B_1 u, \dots, B_m u)$, $u \in C^\infty(\overline{G})$, considered as acting from $W_{L,\{B_j\}}^{2m+s,p}(G)$ into $K_{(0,2m+s-m_j-1/p,p)}$, establishes a homeomorphism between these spaces.*

Replacing, in the argument used for the proof of Theorem 3.1 of [2], the space $W_\gamma^{2m}(G)$ by $W_p^{2m+s}(G)$, it is easy to see that the norm (6) is equivalent to the norm

$$\|u\|_{W_L^{2m+s,p}(G)} = \|u\|_{2m+s,p} + \|Lu\|_{0,p}; \quad (7)$$

therefore in Theorem 4 one may replace $W_{L,\{B_j\}}^{2m+s,p}(G)$ by $W_L^{2m+s,p}(G)$ —the completion of the set $C^\infty(\overline{G})$ with respect to the norm (7), and the assertion of Theorem 4 coincides with the assertion (for integral s) of Theorem 5.4 of [2], proved under the additional assumption of uniqueness of the Dirichlet problem for the equation $Lu = f$.

With the aid of analogous arguments one can obtain from Theorem 3 and Lemmas 1, 2 (for integral s) the homeomorphism theorem 6.16 of [10].

6. Since Theorem 1 is also valid for nonintegral s [5], with the aid of Lemmas 1, 2 it is easy to obtain, for such s , assertions analogous to those proved above. Analogues of the theorems established are valid for operators generated by elliptic, in the sense of Petrovskii, systems of equations and normal boundary conditions [7]. They are also valid for operators generated by elliptic equations or systems with discontinuous coefficients and general boundary conditions and conjugation conditions [5, 7].

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