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1968

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Abstract

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UDC 530.12

PHYSICS

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EQUATION OF MOTION IN AN INTERNAL CENTRALLY SYMMETRIC FIELD IN GENERAL RELATIVITY

(Presented by Academician Ya. B. Zel'dovich, 27 XII 1967)

Let us consider a convenient form of the equations of motion in an internal centrally symmetric field in general relativity. As we shall see, the equations take their simplest form if we take r and the second coordinate m , which plays the role of a Lagrangian coordinate, as the independent variables.

Let us take the metric of the field in the usual form

$$-ds^2 = -c^2 dt^2 e^\nu + dr^2 e^\lambda + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$

Then the basic equations obtained from the conservation laws,

$$T_{i;k}^k = [(p + \varepsilon)u_i u^k + \delta_i^k p]_{;k} = [(p + \varepsilon)u_i u^k]_{;k} + p_{;i} = 0$$

will have the form ⁽¹⁾

$$\frac{1}{c^2 \theta^2} [A u_t + u u_r] - \frac{\omega^2}{c^2} \left[(\ln v)_r + \frac{A u}{c^2} (\ln v)_t \right] = \frac{1}{2u} [A \lambda_t + u \lambda_r] + \frac{\theta^2 T \sigma_r}{W}; \quad (1)$$

$$- [A (\ln v)_t + u (\ln v)_r] + \frac{1}{\theta^2} \left(u_r + \frac{A u}{c^2} u_t \right) + \frac{2u}{r} = \frac{u}{2} \left(\lambda_r + \frac{A u}{c^2} \lambda_t \right); \quad (2)$$

$$A \sigma_t + u \sigma_r = 0. \quad (3)$$

Here $A = e^{(\lambda-\nu)/2}$; $u = A dr/dt$; $\theta^2 = 1 - u^2/c^2$; $W = (p + \varepsilon)v$, u is the 3-velocity; p is the pressure; v is the specific volume; $\varepsilon = \rho c^2$ is the energy density; W is the heat content; $\omega^2/c^2 = -(\partial \ln W / \partial \ln v)_\sigma$; ω is the sound velocity; σ is the entropy; T is the temperature.

The field equations, obtained from the equations $R_i^k - \frac{1}{2}\delta_i^k R = \chi T_i^k$, have the form

$$(re^{-\lambda})_r = 1 - \frac{\chi r^2}{\theta^2} \left(\varepsilon + p \frac{u^2}{c^2} \right); \quad (4)$$

$$Ar(e^{-\lambda})_t = \frac{\chi ur^2}{\theta^2} (\varepsilon + p); \quad (5)$$

$$(1 + rv_r)e^{-\lambda} = 1 + \frac{\chi r^2}{\theta^2} \left(p + \varepsilon \frac{u^2}{c^2} \right); \quad (6)$$

$$\frac{1}{2}e^{-\lambda} \left[\left(v_{rr} + \frac{1}{2}v_r^2 + \frac{1}{2}(v - \lambda)_r - \frac{v_r \lambda_r}{2} \right) - \frac{A}{c^2} (A\lambda_t)_t \right] = \chi p. \quad (7)$$

There must be 5 independent equations in all. They determine p , u , σ , λ , and v ; at the same time, the equation of state of the medium $p = p(\sigma, v)$ must be specified, and the thermodynamic equation $d(\varepsilon v) = T d\sigma - p dv$ and the identity $\partial(p, v)/\partial(T, \sigma) = 1$ must be used.

After certain transformations, we write 2 independent equations of the system (4)-(7) in the form

$$A\lambda_t + u(\lambda_r + (e^\lambda - 1)/r + \chi pre^\lambda) = 0, \quad (8)$$

$$A(1 + u^2/c^2)\lambda_t + u(\lambda + v)_r = 0. \quad (9)$$

Considering the system of equations (1), (2), (3), (8), we see that these equations do not contain v . Consequently, it is necessary to investigate a system of only 4 equations.

Let us pass from the independent variables $(t; r)$ to the variables $(m; r)$, where

$$\varkappa \left(\frac{\partial m}{\partial r} \right)_t = \varkappa \frac{r^2}{\theta^2} \left(\varepsilon + p \frac{u^2}{c^2} \right) = 1 - (re^{-\lambda})_r, \quad (10)$$

whence

$$m = \int_0^r \frac{r^2}{\theta^2} \left(\varepsilon + p \frac{u^2}{c^2} \right) dr = \frac{r}{\varkappa} (1 - e^{-\lambda}). \quad (11)$$

Equation (18) in the variables (m, r) takes the form

$$\left(A - u \frac{\partial t}{\partial m}\right) \frac{\partial \lambda}{\partial m} + u \frac{\partial t}{\partial m} \left(\frac{\partial \lambda}{\partial r} + \frac{e^\lambda - 1}{r} + \kappa p r e^\lambda\right) = 0. \quad (12)$$

Equation (10) takes the form

$$\frac{\partial t}{\partial m} \frac{r^2}{\theta^2} \left(\varepsilon + p \frac{u^2}{c^2}\right) + \frac{\partial t}{\partial m} = 0. \quad (13)$$

Equation (11), which we write in the form $e^{-\lambda} = 1 - \kappa m/r$, determines $\partial e^{-\lambda}/\partial m = -\kappa/r$, $\partial e^{-\lambda}/\partial r = \kappa m/r^2$, or

$$\frac{\partial \lambda}{\partial m} = \frac{\kappa}{r} e^\lambda, \quad \frac{\partial \lambda}{\partial r} = -\frac{\kappa m}{r^2} e^\lambda = -\frac{m}{r} \frac{\partial \lambda}{\partial m}, \quad (14)$$

therefore (12) passes into the equation

$$A - u \frac{\partial t}{\partial r} + u \frac{\partial t}{\partial m} p r^2 = 0. \quad (15)$$

Hence, from (13) we shall have

$$u \partial t / \partial m = -A \theta^2 / r^2 (p + \varepsilon), \quad u \partial t / \partial r = A (\varepsilon + p u^2 / c^2) / (p + \varepsilon). \quad (16)$$

Equations (1), (2), (3) take, in the coordinates (m, r) , the form

$$\begin{aligned} & \frac{u}{c^2 \theta^2} (u_r - p r^2 u_m) - \frac{\omega^2}{c^4} [(\ln v)_r + \varepsilon r^2 (\ln v)_m] = \\ & = \frac{1}{2} (\lambda_r - m^2 \lambda_m) + \frac{T}{W} \left[\theta^2 \sigma_r + r^2 \left(\varepsilon + p \frac{u^2}{c^2} \right) \sigma_m \right], \\ & -u [(\ln v)_r - p r^2 (\ln v)_m] + \frac{1}{\theta^2} [u_r + \varepsilon r^2 u_m] + \frac{2u}{r} = \frac{u}{2} (\lambda_r + \varepsilon r^2 \lambda_m), \\ & \sigma_r - p r^2 \sigma_m = 0. \end{aligned}$$

Transforming the equations, using (14), we shall have

$$\begin{aligned} & \frac{1}{2 c^2 \theta^2} [u_r^2 - p r^2 u_m^2] - \frac{\omega^2}{c^2} [(\ln v)_r + \varepsilon r^2 (\ln v)_m] + \\ & + \frac{\kappa m / r + p r^2}{2(r - \kappa m)} = \frac{r^2 T \sigma_m}{v} = \frac{T \sigma_r}{p v}; \end{aligned} \quad (17)$$

$$\begin{aligned}
 & - [(\ln v)_r - pr^2(\ln v)_m] + \frac{1}{2\theta^2} [(\ln u^2)_r + \varepsilon r^2(\ln u^2)_m] + \\
 & + \frac{2}{r} + \frac{\chi m/r - \varepsilon r^2}{2(r - \chi m)} = 0.
 \end{aligned} \tag{18}$$

As a result we have arrived at a system of only 3 quasilinear equations of the first order.

For isentropic processes, when $\sigma = \text{const}$, the problem reduces to a system of 2 equations.

From equation (16), eliminating A , we shall have

$$\frac{r^2}{\theta^2} \left(\varepsilon + p \frac{u^2}{c^2} \right) \frac{\partial t}{\partial m} + \frac{\partial t}{\partial r} = 0; \tag{19}$$

knowing $\varepsilon = \varepsilon(r, m)$, $u = u(r, m)$ (for a given equation of state), one can (formally) find $t = t(r, m)$, which as a result (again formally) makes it possible to determine $u = u(t, r)$, $\varepsilon = \varepsilon(t, r)$, and, finally, from any

from equation (16) we find

$$\begin{aligned}
 A = e^{(\lambda - \nu)/2} &= -\frac{r^2}{\theta^2} (\varepsilon + p) u \frac{\partial t}{\partial m}; \quad \text{since } e^\lambda = \frac{1}{1 - \chi m/r}, \quad \text{then} \\
 e^{\nu/2} &= \frac{1}{\sqrt{1 - \frac{\chi m}{r}}} \frac{A^2}{r^2 u \frac{\partial t}{\partial m} (p + \varepsilon)},
 \end{aligned} \tag{20}$$

which completely solves the self-consistent problem of finding $u, \varepsilon, \lambda, \nu$ for centrally symmetric motions.

Let us now write the characteristic equations of the system of equations (17), (18). Along the lines

$$m' + pr^2 = 0, \quad \sigma' = 0; \tag{21}$$

here, for example, $\sigma' = d\sigma/dr$, $u' = du/dr$, $m' = dm/dr$, $(\ln \nu)' = d \ln \nu / dr$, etc. Further, expanding the corresponding determinants, we find that along the lines

$$(\varepsilon r^2 - m') = \pm \frac{u}{\omega} (pr^2 + m') \tag{22}$$

the relations

$$(pr^2 + m') \left[\left(\frac{\omega}{c} \pm \frac{u}{c} \right) \left(\frac{\omega}{c} (\ln \nu)' \mp \frac{u'}{c\theta^2} \right) \right] \mp \frac{2u\omega}{c^2 r} - \frac{\chi}{2(r - \chi m)} \left[\frac{m}{r} \left(1 \pm \frac{u\omega}{c^2} \right) + r^2 \left(p \mp \frac{u\omega}{c^2} \varepsilon \right) \right] = \frac{\sigma' T r^2}{\nu}. \quad (23)$$

In the case $p = 0$, the basic equations (17) and (18) are greatly simplified and are integrated directly (Tolman's problem (2)).

The system of equations (1), (2), (3), and (8) also has a solution for the ultra-relativistic approximation, when

$$u/c = 1 - 2\Delta, \quad \Delta \ll 1.$$

In this case, if terms of order Δ^2 are neglected, the equations take the form

$$\begin{aligned} \frac{1}{2\Delta} [A\Delta_\tau + \Delta_r] + \frac{\omega_0^2}{c^2} \left[A(\ln \nu)_\tau + (\ln \nu)_r + \frac{1}{2} [A\lambda_\tau + \lambda_r] \right] &= 0, \\ [A(\ln \nu)_\tau + (\ln \nu)_r] + \frac{1}{2\Delta} (A\Delta_\tau + \Delta_r) - \frac{2}{r} + \frac{1}{2} [A\lambda_\tau + \lambda_r] &= 0, \end{aligned} \quad (24)$$

$$A\sigma_\tau + \sigma_r = 0, \quad A\lambda_\tau + \lambda_r + \frac{e^\lambda - 1}{r} + \chi pre^\lambda = 0,$$

where $\tau = ct$, $\omega_0^2/c^2 = k - 1$.

It is more convenient to write the first two equations of this system in the form

$$\begin{aligned} A(\ln \nu)_\tau + (\ln \nu)_r &= \frac{2}{(2 - k)r}, \\ \frac{1}{2\Delta} [A\Delta_\tau + \Delta_r] + \frac{2(k - 1)}{(2 - k)r} &= \frac{1}{2} \left[\frac{e^\lambda - 1}{r} + \chi pre^\lambda \right]. \end{aligned} \quad (25)$$

Let us pass to the independent variables (ν, r) ; then the last two equations of system (24) and the second equation (25), after eliminating the quantity

$$A - \frac{\partial t}{\partial r} = \frac{2\nu}{(2 - k)r} \frac{\partial t}{\partial \nu} \quad (26)$$

(the first equation (25)), take the form

$$\begin{aligned}\frac{2\nu}{2-k}\frac{\partial\sigma}{\partial\nu} + r\frac{\partial\sigma}{\partial r} &= 0, & \frac{2\nu}{2-k}\frac{\partial\lambda}{\partial\nu} + r\frac{\partial\lambda}{\partial r} + e^\lambda - 1 + \chi pr^2 e^\lambda &= 0, \\ \frac{1}{2\Delta} \left[\frac{2\nu}{2-k}\frac{\partial\Delta}{\partial\nu} + r\frac{\partial\Delta}{\partial r} \right] + \frac{2(k-1)}{2-k} &= \frac{1}{2} [e^\lambda - 1 + \chi pr^2 e^\lambda].\end{aligned}\quad (27)$$

We now transform equation (5) to the coordinates (ν, r) ; as a result we shall have

$$\frac{\partial\tau}{\partial\nu} \left[r\frac{\partial\lambda}{\partial r} + e^\lambda - 1 - \frac{\chi r^2}{4\Delta}(\varepsilon + p)e^\lambda \right] - r\frac{\partial\tau}{\partial r}\frac{\partial\lambda}{\partial\nu} = 0. \quad (28)$$

The solution of the resulting system of equations is carried out as follows. From the first equation of system (27) we obtain

$$\sigma = F_1(r, \nu^{-(2-k)/2}) = \text{const} \cdot p\nu^k. \quad (29)$$

Next we solve the second equation (29) and determine

$$(1 - e^{-\lambda}) = \frac{1}{r}F_2(r\nu^{-(2-k)/2}) + \frac{2-k}{5k-6}\varkappa r^2 p. \quad (30)$$

Further, from the last equation (27) we find

$$\Delta = e^{-\lambda}F_3(r\nu^{-(2-k)/2})r^{-4(k-1)/(2-k)}. \quad (31)$$

From equation (28) we find $\tau = \tau(r, \nu)$, and, finally, equation (26) determines $v = v(r, \nu)$, since

$$A = e^{(\lambda-\nu)/2} = \frac{\partial\tau}{\partial r} + \frac{\partial\tau}{\partial\nu}\frac{2v}{(2-k)r}, \quad \text{whence} \quad e^{-\nu/2} = e^{-\lambda/2}\frac{1}{r} \left[r\frac{\partial\tau}{\partial r} + \frac{\partial\tau}{\partial\nu}\frac{2v}{(2-k)} \right].$$

Detailed calculations here are meaningful only when solving particular problems with already specified functions F_1, F_2, F_3 . As a result we obtain exact asymptotic solutions depending on 5 arbitrary functions.

In conclusion, let us make limiting transitions. For $\varkappa = 0$ we shall have the equations

$$\frac{1}{2c^2\theta^2}[u_r^2 - pr^2u_m^2] - \frac{\omega^2}{c^2}\{(\ln\nu)_r + \varepsilon r^2(\ln\nu)_m\} = \frac{r^2T\sigma_m}{\nu} = \frac{T\sigma_r}{p\nu}; \quad (32)$$

$$-[(\ln \nu)_r - pr^2(\ln \nu)_m] + \frac{1}{2\theta^2 u^2} [u_r^2 + \varepsilon r^2 u_m^2] + \frac{2}{r} = 0, \quad (33)$$

which corresponds to special relativity. For $u/c \ll 1$ we obtain the limiting transition to Newton's theory. Indeed, since in this case $\theta = 1$, $\nu\rho = 1$, $u \partial t / \partial r = 1$, $u \partial t / \partial M_0 = -1/4\pi\rho r^2$, we shall have³

$$uu_r + \omega^2[(\ln \rho)_r + 4\pi\rho r^2(\ln \rho)_{M_0}] + GM_0/r = 4\pi r^2 \rho T \sigma_{M_0}, \quad (34)$$

$$u(\ln \rho)_r + u_r + 4\pi\rho r^2 u_{M_0} + 2u/r = 0, \quad \sigma_r = 0. \quad (35)$$

The equations introduced by us are the simplest for analyzing spherically symmetric motions in the general theory of relativity.

Received
25 XII 1967

CITED LITERATURE

¹ S. M. Kolesnikov, K. P. Stanyukovich, PMM, **29**, no. 4 (1965). ² R. C. Tolman, *Relativity, Thermodynamics and Cosmology*, Oxford, 1934. ³ F. A. Baum, S. A. Kaplan, K. P. Stanyukovich, *Introduction to Cosmic Gas Dynamics*, ch. VIII, § 23, Moscow, 1958.

Note: Figure translations are in progress. See original paper for figures.

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