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Abstract

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MATHEMATICS

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ON QUASILINEAR EQUATIONS IN A BANACH SPACE

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1. In a Banach space E we consider the problem

$$v'(t) = A(t)v(t) + f[t, v(t)] \quad (0 \leq t \leq T), \quad v(0) = v_0. \quad (1)$$

A solution of this problem is a function $v(t)$ satisfying (1) and such that the functions $v'(t)$ and $A(t)v(t)$ are continuous on $[0, T]$. It is assumed that $A(t)$, for each t in $[0, T]$, is the generating operator of a strongly continuous contraction semigroup. The domain D of the operator $A(t)$ does not depend on t , and the operator-function $A(t)A^{-1}(0)$ is strongly continuously differentiable. If the operator A does not depend on t , then it is assumed only that it generates a strongly continuous semigroup.

In work ⁽¹⁾ the unique local solvability of problem (1) was proved under the assumption that the derivatives f'_t and f'_v exist for all t in $[0, T]$ and v in E and satisfy a Lipschitz condition with respect to v . In ⁽²⁾ this fact is established under the assumption of mere continuity of the derivatives f'_t and f'_v . A strengthening of the result from ⁽¹⁾ in another direction was obtained in ⁽³⁾, where it is assumed that the operator f acts (with respect to v) not from E into E , but only from D into E . This makes it possible to consider also unbounded nonlinear operators f .

In work ⁽¹⁾ generalized solutions of problem (1) were also studied, and the existence of at least one generalized solution was proved under the assumption that the operator f is the sum of a Lipschitz operator and a completely continuous operator. Below, in § 2, a theorem is given which generalizes in this respect the result of ⁽¹⁾ for solutions of problem (1), and not only for its generalized solutions. In § 3 an analogous theorem is given for second-order differential equations.

2. **Theorem 1.** Let the operator $f(t, v)$ act from $[0, T] \times D$ into E . Here D is regarded as a Banach space with norm $\|v\|_D = \|A(0)v\|_E$. Let the operator $f(t, v)$ have derivatives $f'_t(t, v)$ and $f'_v(t, v)$. Let $f'_t(t, v) = f_{1t}(t, v) + f_{2t}(t, v)$. Each of the operators $f_{it}(t, v)$ is continuous jointly in the variables. The operator $f_{1t}(t, v)$ satisfies the Lipschitz condition

$$\|f_{1t}(t, v_1) - f_{1t}(t, v_2)\|_E \leq C(R)\|v_1 - v_2\|_D, \quad \|v_i\|_D \leq R.$$

For any fixed t , the operator $f_{2t}(t, v)$ maps every set bounded in D into a set compact in E .

Let the Fréchet derivative $f'_v(t, v)$, which for each fixed t and v is a linear continuous operator acting from D into E , admit a continuous extension to a linear continuous operator acting in E . We shall retain the same notation $f'_v(t, v)$ for this extension. Let $f'_v(t, v) = f_{1v}(t, v) + f_{2v}(t, v)$. For any fixed h in E , the operators $f_{iv}(t, v)$ (from $[0, T] \times D$ into E) are continuous jointly in the variables t and v . The operator $f_{1v}(t, v)h$ satisfies the Lipschitz condition

$$\|[f_{1v}(t, v_1) - f_{1v}(t, v_2)]h\|_E \leq C(R)\|h\|_E\|v_1 - v_2\|_D, \quad \|v_i\|_D \leq R.$$

For any fixed t , the operator $f_{2v}(t, v)h$ maps every set bounded in D into a set compact in E .

Let v_0 belong to D . Then problem (1) has a unique solution, defined on some segment $[0, t_0] \subset [0, T]$.

In the case where $f_{2t} \equiv 0$ and $f_{2v} \equiv 0$, Theorem 1 was proved in ⁽³⁾. Here, in order to shorten the exposition, its proof is given in the case where $A(t) \equiv A$, $f(t, v) \equiv f(v)$, $f'(v) = f'_{2v}(v)$. Thus, let $v(t)$ be a solution of the problem

$$v'(t) = Av(t) + f[v(t)], \quad v(0) = v_0. \quad (2)$$

Then

$$v(t) = \exp\{tA\}v_0 + \int_0^t \exp\{(t-s)A\}f[v(s)] ds. \quad (3)$$

From (3) it follows that

$$\begin{aligned} v'(t) = & \exp\{tA\}Av_0 + \exp\{tA\}f(v_0) + \\ & + \int_0^t \exp\{(t-s)A\}f'[v(s)]v'(s) ds. \end{aligned} \quad (4)$$

Consider the integral equation

$$z(t) = \exp\{tA\}Av_0 + \exp\{tA\}f(v_0) + \int_0^t \exp\{(t-s)A\}f'[v(s)]z(s) ds. \quad (5)$$

Its continuous solution is unique and has the form

$$z(t) = \exp\{tA\}Av_0 + \exp\{tA\}f(v_0) + \int_0^t \exp\{(t-s)A\}\varphi[s, v(s)] ds, \quad (6)$$

where

$$\begin{aligned} \varphi[t, v(t)] &= f'[v(t)]z[t, v(t)], \\ z[t, v(t)] &= \sum_{n=0}^{+\infty} z_n[t, v(t)], \\ z_0[t, v(t)] &= \exp\{tA\}Av_0 + \exp\{tA\}f(v_0), \\ z_{n+1}[t, v(t)] &= \int_0^t \exp\{(t-s)A\}z_n[s, v(s)] ds. \end{aligned} \quad (7)$$

From (4) it then follows that

$$v'(t) = \exp\{tA\}Av_0 + \exp\{tA\}f(v_0) + \int_0^t \exp\{(t-s)A\}\varphi[s, v(s)] ds. \quad (8)$$

Finally, integrating (8), we obtain

$$\begin{aligned} v(t) &= \exp\{tA\}v_0 + [\exp\{tA\} - I]A^{-1}f(v_0) + \\ &+ \int_0^t [\exp\{(t-s)A\} - I]A^{-1}\varphi[s, v(s)] ds. \end{aligned} \quad (9)$$

Making in equation (9) the substitution $w(t) = Av(t)$, we reduce the problem to finding a continuous solution in E of the equation

$$w(t) = \exp\{tA\}Av_0 + [\exp\{tA\} - I]f(v_0) + \int_0^t [\exp\{(t-s)A\} - I]\varphi[s, A^{-1}w(s)] ds. \quad (10)$$

We shall regard (10) as an operator equation in the space $C([0, t_0], E)$ of continuous functions $w(t)$ defined on $[0, t_0]$ with values ...

chains in E . Write it in the form $w = Kw$. For sufficiently small t_0 the operator K maps some ball of the space C into itself and is continuous in C . From Mazur's theorem it follows that the operator K transforms any bounded set of functions $N = \{w(t)\}$ in C into a set of such functions $KN = \{Kw(t)\}$ that the set of their values for each fixed t is compact in E .

Finally, let us prove that KN is a set of equicontinuous functions. For this, with the aid of the Schauder projection operator (see, for example, ⁽⁴⁾) P_n , we construct a sequence of operators K_n converging uniformly on N to K . Finally, P_n (and correspondingly K_n) is slightly deformed so that the basis vectors belong to D . Then $\tilde{K}_n N$, for the resulting $\tilde{K}_n$, will be a set of equicontinuous functions. Thus the Schauder principle is applicable to the equation $w = Kw$. Consequently, problem (2) has at least one solution $v(t)$, defined on some segment $[0, t_0]$.

The smoothness of the operator f makes it possible to prove uniqueness, although it does not allow the problem to be reduced to the contraction mapping principle. If the operator $f'(v)$ were the sum of a Lipschitz operator $f_1(v)$ and a completely continuous operator $f_2(v)$, then the equation with the completely continuous operator could be reached in a somewhat different way. From equality (4) we pass to the equality

$$v(t) = \exp\{tA\}v_0 + [\exp\{tA\} - I]A^{-1}f(v_0) + \int_0^t [\exp\{(t-s)A\} - I]A^{-1}f'[v(s)]v'(s) ds, \quad (11)$$

and then consider the system, following from (4) and (11),

$$\begin{aligned}
 z(t) &= \exp\{tA\}Av_0 + \exp\{tA\}f(v_0) + \\
 &\quad + \int_0^t \exp\{(t-s)A\}(f_1[A^{-1}w(s)] + f_2[v(s)])z(s) ds, \\
 w(t) &= \exp\{tA\}Av_0 + [\exp\{tA\} - I]f(v_0) + \\
 &\quad + \int_0^t [\exp\{(t-s)A\} - I](f_1[A^{-1}w(s)] + f_2[v(s)])z(s) ds,
 \end{aligned} \tag{12}$$

where $z(t) = v'(t)$, $w(t) = Av(t)$. System (12) can be solved uniquely, since $z(t)$ enters linearly, while $w(t)$ enters under the sign of the Lipschitz operator f_1 .

3. For the investigation of quasilinear equations of higher orders one may pass to a system of first-order equations, and then apply results ⁽¹⁻³⁾ or Theorem 1. Thus one may proceed in the investigation of the problem

$$\begin{aligned}
 v''(t) + A(t)v(t) &= f[t, v(t), v'(t)] \quad (0 \leq t \leq T), \\
 v(0) &= v_0, \quad v'(0) = v'_0
 \end{aligned} \tag{13}$$

in a real Hilbert space H with a positive definite self-adjoint operator $A(t)$. Its solution is a function $v(t)$ satisfying (13) and such that the functions $v''(t)$ and $A(t)v(t)$ are continuous on $[0, T]$. This problem, as S. G. Krein showed, can be reduced to a system of first-order equations with a matrix operator satisfying the conditions of item 1, but for this one has to consider the problem in a complex Hilbert space.

If in problem (13) the operator $A(t) = A(0) = A$, then problem (13) is equivalent to the problem

$$v(t) = \cos A^{1/2}t v_0 + A^{-1/2} \sin A^{1/2}t v'_0 + \int_0^t A^{-1/2} \sin A^{1/2}(t-s) f[s, v(s), v'(s)] ds \tag{14}$$

in the same real Hilbert space H . The methods developed in (5) make it possible to reduce also the problem with a variable operator $A(t)$ to an analogous equation.

Theorem 2. Let the domain of definition D of the operator $A(t)$ not depend on t , and let the operator-function $A(t)A^{-1}(0)$ be strongly continuously differentiable. Let the operator $f(t, v, w)$ act from $[0, T] \times D \times D_{1/2}$ into H . Here $D_{1/2}$ denotes the domain of definition of the operator $A^{1/2}(0)$ with norm $\|w\|_{D_{1/2}} = \|A^{1/2}(0)w\|_H$. Let the operator $f(t, v, w)$ have the derivatives $f'_t(t, v, w)$, $f'_v(t, v, w)$, $f'_w(t, v, w)$. Let $f'_t(t, v, w) = f_{1t}(t, v, w) + f_{2t}(t, v, w)$. Each of the operators $f_{it}(t, v, w)$ is continuous in the aggregate of variables. The operator $f_{1t}(t, v, w)$ satisfies the Lipschitz condition

$$\|f_{1t}(t, v_1, w_1) - f_{1t}(t, v_2, w_2)\|_H \leq C(R)(\|v_1 - v_2\|_D + \|w_1 - w_2\|_{D_{1/2}}),$$

$$\|v_i\|_D \leq R, \quad \|w_i\|_{D_{1/2}} \leq R.$$

The operator $f_{2t}(t, v, w)$, for any fixed t , maps every set bounded in $D \times D_{1/2}$ into a set compact in H . Let the Fréchet derivative $f'_v(t, v, w)$, which for each fixed t, v, w is a linear continuous operator acting from D into H , admit a continuous extension to a linear continuous operator acting from $D_{1/2}$ into H . We retain the same notation $f'_v(t, v, w)$ for this extension. Let $f'_v(t, v, w) = f'_{1v}(t, v, w) + f'_{2v}(t, v, w)$. For any fixed $h \in D_{1/2}$, the operators $f'_{iv}(t, v, w)h$ (from $[0, T] \times D \times D_{1/2}$ into H) are continuous in the aggregate of variables t, v, w . The operator $f'_{1v}(t, v, w)h$ satisfies the Lipschitz condition

$$\|[f'_{1v}(t, v_1, w_1) - f'_{1v}(t, v_2, w_2)]h\|_H \leq$$

$$\leq C(R)\|h\|_{D_{1/2}}(\|v_1 - v_2\|_D + \|w_1 - w_2\|_{D_{1/2}}), \quad \|v_i\|_D \leq R, \quad \|w_i\|_{D_{1/2}} \leq R.$$

The operator $f_{2v}(t, v, w)h$, for any fixed t , maps every set bounded in $D \times D_{1/2}$ into a set compact in H .

Let the Fréchet derivative $f'_w(t, v, w)$, which for each fixed t, v, w is a linear continuous operator acting from $D_{1/2}$ into H , admit a continuous extension to a linear continuous operator acting in H . We retain the same notation $f'_w(t, v, w)$ for this extension. Let $f'_w(t, v, w) = f'_{1w}(t, v, w) + f'_{2w}(t, v, w)$. For any fixed h from H , the operators $f'_{iw}(t, v, w)$ (from $[0, T] \times D \times D_{1/2}$ into H) are continuous in the aggregate of variables t, v, w . The operator $f'_{1w}(t, v, w)$ satisfies the Lipschitz condition

$$\|[f'_{1w}(t, v_1, w_1) - f'_{1w}(t, v_2, w_2)]h\|_H \leq$$

$$\leq C(R)\|h\|_H(\|v_1 - v_2\|_D + \|w_1 - w_2\|_{D_{1/2}}), \quad \|v_i\|_D \leq R, \quad \|w_i\|_{D_{1/2}} \leq R.$$

The operator $f_{2w}(t, v, w)h$, for any fixed t , maps every set bounded in $D \times D_{1/2}$ into a set compact in H . Let $v_0 \in D$, $v'_0 \in D_{1/2}$.

Then problem (13) has a unique solution, defined on some segment $[0, t_0] \subset [0, T]$.

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