

ON THE APPLICATION OF THE DIFFERENCE METHOD TO THE SOLUTION OF NONLINEAR ELLIPTIC EQUATIONS

N. N. Gudovič

1968

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196801.54476>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.949+517.944/.947

MATHEMATICS

N. N. GUDOVICH

ON THE APPLICATION OF THE DIFFERENCE METHOD TO THE SOLUTION OF NONLINEAR ELLIPTIC EQUATIONS

(Presented by Academician M. A. Lavrent'ev on 26 V 1967)

The solvability of the Dirichlet problem for nonlinear elliptic equations has been investigated by a number of authors (see ⁽¹⁻⁴⁾). In the present note, under the assumptions of ⁽³⁾, existence theorems are proved by means of the grid method, and the question of strong and weak convergence of approximate solutions to the exact solution is considered.

1°. Statement of the problem. Let Ω be a bounded domain in R^n satisfying the cone condition. Denote by N the number of partial derivatives in R^n of order $\leq m$, and let $\{A_\alpha(x, \vec{\xi})\}$ (α is a differentiation multi-index, $|\alpha| \leq m$) be a family of N functions acting from $\Omega \times R^N$ into R and satisfying the Carathéodory condition. Suppose that:

- 1) for every element $\vec{\xi}(x) = \{\xi_\alpha(x)\}$ of $\mathcal{L}_p^N(\Omega) = \mathcal{L}_p(\Omega) \times \dots \times \mathcal{L}_p(\Omega)$ ($p > 1$) the inclusion

$$A_\alpha(x, \vec{\xi}(x)) \in \mathcal{L}_{p'}(\Omega);$$

- 2) the form

$$a(\vec{\xi}, \vec{\eta}) = \sum_{|\alpha| \leq m} \int_{\Omega} A_\alpha(x, \vec{\xi}) \eta_\alpha dx$$

satisfies the coercivity condition:

$$\lim_{\|\vec{\xi}\|_{L_p^N(\Omega)} \rightarrow \infty} a(\vec{\xi}, \vec{\xi}) / \|\vec{\xi}\|_{L_p^N(\Omega)} = \infty \quad (1,1)$$

$$\left(\|\vec{\xi}\|_{L_p^N(\Omega)} = \left[\sum_{|\alpha| \leq m} \|\xi_\alpha\|_{L_p(\Omega)}^p \right]^{1/p} \right)$$

on the set of collections $\vec{\xi} = \{\xi_\alpha\} \in \mathcal{L}_p^N(\Omega)$ possessing the following subordination property: for every α ($|\alpha| \leq m-1$) there exists α' ($|\alpha'| = |\alpha| + 1$) such that

$$\|\xi_\alpha\|_{L_p(\Omega)} \leq \text{diam } \Omega \cdot \|\xi_{\alpha'}\|_{L_p(\Omega)}; \quad (1,2)$$

- 3) let $K > 0$ be an arbitrary constant and $E_N(K)$ the set of collections $\vec{\xi} = \{\xi_\alpha\}$ of N numbers such that $\sum_{|\alpha| \leq m-1} |\xi_\alpha| \leq K$; then for every x from a set Ω' ($\Omega' \subset \Omega$) of full measure,

$$\lim_{\substack{\sum_{|\alpha|=m} |\xi_\alpha| \rightarrow \infty \\ \vec{\xi} \in E_N(K)}} \sum_{|\alpha|=m} A_\alpha(x, \vec{\xi}) \xi_\alpha / \left[\sum_{|\alpha|=m} |\xi_\alpha| + \left(\sum_{|\alpha|=m} |\xi_\alpha| \right)^{p-1} \right] = \infty; \quad (1,3)$$

- 4) for any x from Ω' and any sets $\vec{\xi} = \{\xi_\alpha\}$, $\vec{\eta} = \{\eta_\alpha\}$, such that $\vec{\xi} \neq \vec{\eta}$ and $\xi_\alpha = \eta_\alpha$ for $|\alpha| \leq m-1$, one has

$$\sum_{|\alpha|=m} (A_\alpha(x, \vec{\xi}) - A_\alpha(x, \vec{\eta}))(\xi_\alpha - \eta_\alpha) \geq a_0 \left(\sum_{|\alpha|=m} |\xi_\alpha - \eta_\alpha|^p \right) \quad (1,4')$$

or

$$\sum_{|\alpha|=m} (A_\alpha(x, \vec{\xi}) - A_\alpha(x, \vec{\eta}))(\xi_\alpha - \eta_\alpha) > 0. \quad (1,4'')$$

Let $\mathbf{f} = \{f_\alpha\} \in \mathcal{L}_{p'}^N(\Omega)$ be a set of N functions from $\mathcal{L}_{p'}(\Omega)$

$$\left(\|\mathbf{f}\|_{\mathcal{L}_{p'}^N(\Omega)} = \left[\sum_{|\alpha| \leq m} \|f_\alpha\|_{\mathcal{L}_{p'}(\Omega)}^{p'} \right]^{1/p'} \right).$$

Consider the following problem: in the subspace $P \subset \mathcal{L}_p^N(\Omega)$ of elements of the form $\mathbf{v} = \{D^\alpha v\}$ ($v \in \dot{W}_p^m(\Omega)$), find a set $\mathbf{u} = \{D^\alpha u\}$ such that for any $\mathbf{w} = \{D^\alpha w\} \in P$

$$a(\mathbf{u}, \mathbf{w}) \equiv \sum_{|\alpha| \leq m} \int_\Omega A_\alpha(x, \mathbf{u}) D^\alpha w \, dx = \sum_{|\alpha| \leq m} \int_\Omega f_\alpha D^\alpha w \, dx. \quad (1,5)$$

2°. Approximating spaces and operators.

Introduce the following notation: Ω_h is the union of all cells of the mesh lying in Ω ; $\dot{\Omega}_h$ is the union of cells from Ω_h whose centers are at a distance $> mh$ from the boundary of Ω_h ; $\Omega_h^\alpha = \Omega_h \cap [T_n^{i_n} \dots T_1^{i_1} \Omega_h]$ is a domain in R^n ($\alpha = (i_1, \dots, i_n)$ is a multi-index, $|\alpha| \leq m$; T_s is the shift operator in R^n by h ($h > 0$) in the negative direction of the x_s -axis); D_{hs} is the difference operator from $\mathcal{L}_p(\Omega_h^\alpha)$ into $\mathcal{L}_p(\Omega_h^{\alpha'})$ ($\alpha = (i_1, \dots, i_s, \dots, i_n)$, $|\alpha| \leq m-1$; $\alpha' = (i_1, \dots, i_s+1, \dots, i_n)$), defined by the formula

$$(D_{hs}u)(x) = [u(x + he_s) - u(x)]/h,$$

where e_s is the unit vector of the x_s -axis; $D_h^\alpha = D_{h1}^{i_1} \dots D_{hn}^{i_n}$ is an operator from $\mathcal{L}_p(\Omega_h)$ into $\mathcal{L}_p(\Omega_h^\alpha)$; $\overline{\mathcal{L}}_p(\Omega_h^\alpha)$ is the subspace of functions from $\mathcal{L}_p(\Omega)$ that vanish in $\Omega - \Omega_h^\alpha$ and take constant values in the cells of the domain Ω_h^α ; $\overline{D}_h^\alpha$ is the operator from $\overline{\mathcal{L}}_p(\Omega_h)$ into $\overline{\mathcal{L}}_p(\Omega_h^\alpha)$ induced by the operator D_h^α ; $\overline{\mathcal{L}}_p(\dot{\Omega}_h)$ is the set of functions from $\overline{\mathcal{L}}_p(\Omega_h)$ vanishing in $\Omega_h - \dot{\Omega}_h$. Replace problem (1,5) by the approximate one: in the subspace $P_h \subset \mathcal{L}_p^N(\Omega)$ of sets $\mathbf{v}_h = \{\overline{D}_h^\alpha v_h\}$, where $v_h \in \overline{\mathcal{L}}_p(\dot{\Omega}_h)$, find a set $\mathbf{u}_h = \{\overline{D}_h^\alpha u_h\}$ such that for any $\mathbf{w}_h = \{\overline{D}_h^\alpha w_h\} \in P_h$ the equality holds:

$$\sum_{|\alpha| \leq m} \int_{\Omega} A_\alpha(x, \mathbf{u}_h) \overline{D}_h^\alpha w_h dx = \sum_{|\alpha| \leq m} \int_{\Omega} f_\alpha \overline{D}_h^\alpha w_h dx. \quad (2,1)$$

3°. Convergence of the approximate solutions.

Lemma 1. Let $u_h(x) \in \overline{\mathcal{L}}_p(\dot{\Omega}_h)$ vanish in the boundary strip of the domain Ω_h of width h , and let $|\alpha| = 1$. Then

$$\|u_h(x)\|_{\mathcal{L}_p(\Omega)} \leq \text{diam } \Omega \cdot \|\overline{D}_h^\alpha u_h(x)\|_{\mathcal{L}_p(\Omega)}. \quad (3,1)$$

Lemma 2. Let the sequence $\{v_h(x)\}$ ($v_h(x) \in \overline{\mathcal{L}}_p(\Omega_h)$; $h \rightarrow 0$) be such that v_h vanish in the boundary strip of the domain Ω_h of width h and, for any α ($|\alpha| = 1$), $v_h \rightarrow v^*$, $\overline{D}_h^\alpha v_h \rightarrow v_\alpha^*$ weakly in $\mathcal{L}_p(\Omega)$. Then $v^*(x) \in \dot{W}_p^1(\Omega)$ and $D^\alpha v^* = v_\alpha^*$.

Lemma 3. Let the sequence $\{v_h(x)\}$ satisfy the conditions of Lemma 2. Then $v_h(x) \rightarrow v^*(x)$ strongly in $\mathcal{L}_p(\Omega)$.

It follows from Lemma 1 that the elements P_h satisfy the coercivity condition (1,2). But then, using (1,1) and the finite dimensionality of P_h , one can, as in (3), show that for any h problem (2,1) is solvable. Moreover,

from the inequality

$$a(u_h, u_h) = \sum_{|\alpha| \leq m} \int_{\Omega} f_{\alpha} \overline{D}_h^{\alpha} u_h dx \leq \|f\|_{L_{p'}^N(\Omega)} \cdot \|u_h\|_{L_p^N(\Omega)}$$

and condition (1.1), it follows that for the solutions of problem (2.1) the estimates

$$\|u_h\|_{L_p^N(\Omega)} \leq K, \quad K \neq K(h) \quad (3.2)$$

hold.

Therefore there exists a subsequence $u_{h_i} = \{\overline{D}_{h_i}^{\alpha} u_{h_i}\}$ of approximate solutions such that $\overline{D}_{h_i}^{\alpha} u_{h_i}$ converges weakly in $\mathcal{L}_p(\Omega)$, for all α ($|\alpha| \leq m$), to the components u_{α}^* of the vector $\mathbf{u}^* = \{u_{\alpha}^*\}$ from $\mathcal{L}_p^N(\Omega)$. At the same time, Lemma 2 implies the inclusion $\mathbf{u}^* \in P$, and Lemma 3 implies that, for $|\alpha| \leq m-1$,

$$\overline{D}_{h_i}^{\alpha} u_{h_i} \rightarrow u_{\alpha}^* = D^{\alpha} u^* \quad \text{strongly in } \mathcal{L}_p(\Omega). \quad (3.3)$$

Next, let (1.4') be fulfilled. Consider the form

$$a_1(\vec{\xi}, \vec{\eta}, \vec{\zeta}) = \sum_{|\alpha|=m} \int_{\Omega} A_{\alpha}(x, \vec{\xi}, \eta^m) \zeta_{\alpha} dx + \sum_{|\alpha| \leq m-1} \int_{\Omega} A_{\alpha}(x, \vec{\xi}, \xi^m) \zeta_{\alpha} dx,$$

where $\vec{\xi} = \{\xi_{\alpha}\}$ ($|\alpha| \leq m$), $\vec{\xi} = \{\xi_{\alpha}\}$ ($|\alpha| \leq m-1$), $\xi^m = \{\xi_{\alpha}\}$ ($|\alpha| = m$).

The following relations are valid:

$$\lim_{h_i \rightarrow 0} a_1(u_{h_i}, u_{h_i}, v) = \sum_{|\alpha| \leq m} \int_{\Omega} f_{\alpha} D^{\alpha} v dx \quad (v \in \dot{W}_p^m(\Omega)); \quad (3.4)$$

$$\lim_{h_i \rightarrow 0} a_1(u_{h_i}, u_{h_i}, u_{h_i}) = \sum_{|\alpha| \leq m} \int_{\Omega} f_{\alpha} D^{\alpha} u^* dx. \quad (3.5)$$

Moreover, by virtue of (3.2), the functions $A_{\alpha}(x, \tilde{u}_{h_i}, u_{h_i}^m)$ are bounded in $\mathcal{L}_{p'}(\Omega)$ uniformly in i . From this fact, relations (3.3)–(3.5), and the inequality

$$\begin{aligned} & a_1(u_{h_i}, u_{h_i}, u_{h_i} - u^*) - a_1(u_{h_i}, u^*, u_{h_i} - u^*) \geq \\ & \geq a_0 \left[\sum_{|\alpha|=m} \|D^{\alpha} u^* - \overline{D}_{h_i}^{\alpha} u_{h_i}\|_{L_p(\Omega)}^p \right] \end{aligned}$$

it follows that also for $|\alpha| = m$

$$\overline{D}_{h_i}^\alpha u_{h_i} \rightarrow D^\alpha u^* \quad \text{strongly in } \mathcal{L}_p(\Omega).$$

If, instead of (1.4'), condition (1.4'') is fulfilled, then, by arguments close to those used in (3), one can show that

$$A_\alpha(x, \tilde{u}_{h_i}, u_{h_i}^m) \rightarrow A_\alpha(x, \tilde{u}^*, u^{*m})$$

weakly in $\mathcal{L}_{p'}(\Omega)$.

In both cases, passing in (2.1) to the limit as $h_i \rightarrow 0$, we see that u^* satisfies equation (1.5).

Theorem 1. *Let A_α satisfy conditions 1)–3) of Sec. 1 and (1.4') $\{(1.4'')\}$. Then, for any vector $\{f_\alpha\} \in \mathcal{L}_{p'}^N(\Omega)$, problem (1.5) is solvable, and there exists a subsequence of approximate solutions converging strongly $\{\text{weakly}\}$ to the exact solution in $\mathcal{L}_p^N(\Omega)$.*

Let us note that if the solution of problem (1.5) is unique, then the entire sequence of approximate solutions also converges to the exact solution.

4°. Modification of the difference scheme. Suppose, in addition, that the $f_\alpha(x)$ are continuous in Ω and

$$|A_\alpha(x, \vec{\xi}) - A_\alpha(x', \vec{\xi})| \leq \varphi_\alpha(|x - x'|) |g_\alpha(\vec{\xi})|, \quad (4.1)$$

where $\varphi_\alpha(|x - x'|) \rightarrow 0$ as $|x - x'| \rightarrow 0$, and the functions continuous in R^N ,

$$g_\alpha(x) \quad (g_\alpha(\vec{\xi}(x)) \in \mathcal{L}_{p'}(\Omega) \text{ for } \vec{\xi}(x) \in \mathcal{L}_p^N(\Omega))$$

possess, on vectors

(1, 2) with sufficiently large norm $(\|\vec{\xi}(x)\|_{\mathcal{L}_p^N(\Omega)} \geq R_0)$ the property

$$\left(\sum_{|\alpha| \leq m} \|g_\alpha(\vec{\xi})\|_{\mathcal{L}_{p'}(\Omega)}^{p'} \right)^{1/p'} / \frac{a(\vec{\xi}, \vec{\xi})}{\|\vec{\xi}\|_{\mathcal{L}_p^N(\Omega)}} \leq K. \quad (4.2)$$

Put

$$a_h(\vec{\xi}, \vec{\eta}) \equiv \sum_{|\alpha| \leq m} \int_{\Omega} A_{\alpha h}(x, \vec{\xi}) \eta_\alpha dx, \quad (4.3)$$

where $A_{\alpha h}(x, \vec{\xi}) = A_{\alpha}(x, \vec{\xi})$ for $x \in \Omega_h^{\alpha}$, and $A_{\alpha h}(x, \vec{\xi}) = A_{\alpha}(x^{j_1, \dots, j_n}, \vec{\xi})$, if x belongs to the cell of the domain Ω_h^{α} with center $x^{j_1, \dots, j_n}$. We replace (2.1) by the following problem:

$$a_h(\mathbf{u}_h, \mathbf{w}_h) \equiv \sum_{|\alpha| \leq m} \int_{\Omega} A_{\alpha h}(x, \mathbf{u}_h) \bar{D}_h^{\alpha} w_h \, dx = \sum_{|\alpha| \leq m} \int_{\Omega} f_{\alpha h} \bar{D}_h^{\alpha} w_h \, dx. \quad (4.4)$$

In this equation, $f_{\alpha h}$ denotes functions constructed from f_{α} in the same way as $A_{\alpha h}$ is constructed from A_{α} .

Scheme (4.4) is more convenient for computations, since the integrals entering it are expressed as finite sums.

Let us note that from (4.1) there follows the inequality

$$|a(\vec{\xi}, \vec{\eta}) - a_h(\vec{\xi}, \vec{\eta})| \leq c_h \left[\sum_{|\alpha| \leq m} \|g_{\alpha}(\vec{\xi})\|_{\mathcal{L}_{p'}(\Omega)}^{p'} \right]^{1/p'} \|\vec{\eta}\|_{\mathcal{L}^{p^N}(\Omega)}, \quad (4.5)$$

where $c_h \rightarrow 0$ as $h \rightarrow 0$.

Therefore, on sets $(1, 2)$ with $\|\vec{\xi}\|_{\mathcal{L}^{p^N}(\Omega)} \geq R_0$,

$$a_h(\vec{\xi}, \vec{\xi}) / \|\vec{\xi}\|_{\mathcal{L}^{p^N}(\Omega)} \geq (1 - c_{hK}) a(\vec{\xi}, \vec{\xi}) / \|\vec{\xi}\|_{\mathcal{L}^{p^N}(\Omega)}. \quad (4.6)$$

It follows that, for all sufficiently small h , the system (4.4) is solvable, and for its solutions (3.3) is valid. Further, with the aid of the estimates (3.3), (4.5), assertions analogous to Theorem 1 are proved.

The author expresses his gratitude to Prof. S. G. Krein for valuable comments.

Voronezh State University

Received

24 V 1967

CITED LITERATURE

1. I. M. Vishik, DAN, **138**, 518 (1961).
2. F. E. Browder, Proc. Nat. Acad. Sci. U.S.A., **50**, No. 1, 31 (1963).
3. J. Leray, I. L. Lions, Séminaire de Collège de France, No. 1, 1 (1964).
4. Yu. A. Dubinskii, *Nonlinear elliptic and parabolic equations of arbitrary order*, Candidate's dissertation, Moscow Power Engineering Institute, 1965.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.