

ON THE NONEXISTENCE OF MAPPINGS OF CLASS $\backslash(BL^{\{n/2\}}\backslash)$ OF A BALL ONTO A DOMAIN

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Abstract

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MATHEMATICS

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ON THE NONEXISTENCE OF MAPPINGS OF CLASS $BL^{n/2}$ OF A BALL ONTO A DOMAIN

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1. A continuous vector function $y = f(x)$, defined in a domain D of n -dimensional Euclidean space E^n and with values in E^m , belongs to the class of functions $BL_K^{n/2}$ if

$$I(f, D) = \int_D \left[\sum_{i=1}^m \sum_{j=1}^n \left(\frac{\partial f_i}{\partial x_j} \right)^2 \right]^{n/2} dx \leq K < \infty,$$

where dx is the volume element in E^n , and the derivatives are understood in the sense of S. L. Sobolev.

If $f \in BL_K^{n/2}$ for some K , then $f \in BL^{n/2}$. If the function $y = f(x)$ realizes a homeomorphism of the ball $D : |x| < R$ onto a domain Δ , with $f, f^{-1} \in BL_K^{n/2}$, the domain Δ contains the ball $D_1 : |y| < \delta$, and $f(0) = 0$, then the function f belongs to the class of functions $[BL]_K^{n/2}$. If $f \in [BL]_K^{n/2}$ for some K , then $f \in [BL]^{n/2}$.

For $n = m = 1$, the integral $I(f, D)$ gives the variation of the function f on the interval D . For $n = m = 2$, $I(f, D)$ is the usual Dirichlet integral, equal for conformal mappings to twice the measure of the domain $f(D)$. For quasiconformal mappings with bounded measure of the domain $f(D)$, the integral $I(f, D)$ is bounded.

Of fundamental importance in studying the properties of mappings of class $BL^{n/2}$ is the inequality of Theorem 2 of the work ⁽¹⁾. If, instead of $I(f, D)$, one substitutes into this inequality the integral

$$V_f(D) = \int_D \lambda_f^n(x) dx,$$

where

$$\lambda_f(x) = \overline{\lim}_{\Delta x \rightarrow 0} \frac{|f(x + \Delta x) - f(x)|}{|\Delta x|},$$

then it is preserved up to a constant factor, since

$$\lambda_f(x) \leq \left[\sum_{i=1}^m \sum_{j=1}^n \left(\frac{\partial f_i}{\partial x_j} \right)^2 \right]^{1/2} \leq \sqrt{n} \lambda_f(x).$$

The quantity $V_f(D)$ is naturally called the variation of the function over the domain D . The class of functions $BL_K^{n/2}$ is thus a generalization of the class of functions of bounded variation (more precisely, of the class of functions of bounded variation, absolutely continuous inside the interval D). The classes of functions $BL_K^{n/2}$ and $[BL]_K^{n/2}$ have many properties in common with the class of conformal mappings in the plane. The most essential difference is the absence of an analogue of the Riemann theorem for $n \geq 3$. The results given below show that, in the general case for $n \geq 3$, the question of the existence of mappings of a ball onto a domain within the class under consideration is of exceptional difficulty. For $n = m = 2$, mappings of this class were studied by J. Lelong-Ferrand ⁽²⁾ and G. D. Suvorov ⁽³⁾. The study of the spatial case for $n = m = 3$ was initiated in the works ⁽⁴⁻⁷⁾, and was subsequently continued for $n \geq 2$ and arbitrary m in ^(1, 8, 9).

Below we shall use the following notation: $\rho(M_1, M_2)$ is the distance between sets in E^m ; $|x' - x''|$ is the distance between points-

in E^n ; $\overline{M}$ is the closure of the set M in E^n ; $\partial\Delta$ is the boundary of the domain Δ ; $d(M)$ is the diameter of the set M in E^n ; $C_D(f, a, l)$ is the cluster set of the function f at the point a of the boundary of the domain D relative to the curve $l \subset D$, i.e., it is the set of limit points of sequences $\{f(a_m)\}$, $a_m \in l$ ($m = 1, 2, \dots$), $\lim_{m \rightarrow \infty} a_m = a$.

2. For bounded domains $\Delta \subset E^n$, compact with respect to the relative distance ⁽⁵⁾ and belonging to the class of domains $A_1(S)$ (for any sphere S in E^n , every component of the set $S \cap \Delta$ divides the domain Δ into two subdomains), the compactification of the domain by prime ends, analogous to the prime ends of Carathéodory ⁽¹⁰⁾, is a compactum ⁽⁸⁾. The prime ends of this class of domains contain 5 types of prime ends. To the 4 types in Carathéodory's classification there is added a prime end of the 5th type, whose body has a disconnected closed set of principal points.* As the example of domains given below shows, for domains in E^n ($n \geq 3$) all 5 types of prime ends are realized. On the other hand, by the corollary of Theorem 1, by means of a homeomorphism of class $BL^{n/2}$ a ball cannot be mapped onto a domain having prime ends of the 5th type.

Theorem 1. *If the function $y = f(x)$ realizes a homeomorphic mapping of class $BL^{n/2}$ of the ball $D : |x| < 1$ onto a bounded domain Δ , then for every non-tangential curve $l \subset D$ tending to a point $a \in \partial D$, the set $C_D(f, a, l)$ coincides with the set $|e|_1$ of principal points of the prime end $e = f(a)$.*

This theorem is an analogue of the well-known Lindelöf theorem for conformal mappings. For mappings of class BL it was established by J. Lelong-Ferrand ⁽²⁾. The proof of this theorem is based on the inequality from ⁽¹⁾ and the results from ⁽⁸⁾.

Corollary. *There do not exist homeomorphic mappings of class $BL^{n/2}$ of the ball $D : |x| < 1$ onto a bounded domain Δ having prime ends with a disconnected set of principal points.*

We give an example of domains in E^3 , homeomorphic to the ball and compact with respect to the relative distance, which contain prime ends of the 5th type.

Example 1. Consider in E^3 the cube $R = \{x \in E^3 : 0 < x_i < 1 \ (i = 1, 2, 3)\}$ and the sets $T_{mp} = \{x \in R : x_1 = 1/2^m 3^p\} \ (m, p = 1, 2, \dots)$. Let F be an arbitrary closed set on the face $T_0 = \{x \in \bar{R} : x_1 = 0\}$ of the cube R . We construct a domain Δ having a prime end e such that $|e| = T_0$, $|e|_1 = F$. We shall assume that F is an infinite set. For finite F , the constructions are carried out analogously.

Let $\{a_m\} \ (m = 1, 2, \dots)$, $\{a_m\} \subset F$, be some countable set of points of the face T_0 , everywhere dense in F , with the points of $\{a_m\}$ not lying on the edges of the cube R . Put $Q_{mp} = \{x \in T_{mp} : |b_{mp} - x| < r_{mp}\}$, where $b_{mp} = a_m + c_{mp}$, $c_{mp} = (1/2^m 3^p, 0, 0)$, $r_{mp} = \min[1/2^m 3^p, \rho(a_m, \partial T_0)]$, and ∂T_0 is the boundary of T_0 , considered as a plane set (in the plane $x_1 = 0$). Then the set

$$\Delta = R \setminus \left[\bigcup_{m,p=1}^{\infty} (T_{mp} \setminus Q_{mp}) \right]$$

is a bounded domain that is homeomorphic to the ball and compact with respect to the relative distance. Consider a sequence of sections $\{q_n\} \ (n = 1, 2, \dots)$, whose set coincides with the set of circles $\{Q_{mp}\} \ (m, p = 1, 2, \dots)$ and which are numbered in the order of approach to the face T_0 . This sequence of sections defines a prime end e of the domain Δ such that $|e| = T_0$, $|e|_1 = F$. Thus, if

* Let a prime end e of a domain Δ be defined by means of a chain of subdomains $\{g_m\}$. The body of the prime end e is the set $|e| = \bigcap_{m=1}^{\infty} \bar{g}_m$. A point $a \in |e|$ is called a principal point of the prime end e if there exists a chain of sections $\{q_m\}$ defining the prime end e and contracting to the point a .

If F is a disconnected set, then the simple end e of the constructed domain Δ will be of type 5. For $n = 2$ the set of principal points of a simple end is always connected ⁽¹⁰⁾.

3. Let us give a lower estimate for the quantity K for a mapping $y = f(x)$ of class $[BL]_K^{n/2}$ of the ball D onto the domain Δ . This estimate will involve the function $\psi(a, \Delta)$ of the variable a , which is defined as follows. Let $G_a^0(\Delta)$ be the component of the set

$$G_a(\Delta) = \{x \in \Delta : \rho(x, \partial\Delta) > a\},$$

containing the point 0. Consider a point $a \in \Delta \setminus G_a^0(\Delta)$ and define the function $h(a, a)$ of the variable a : $h(a, a) = \inf d(K)$, where the infimum is taken over all continua $K \subset \Delta$ separating the points a and 0 in Δ and such that the sets $K \cap G_a^0(\Delta)$ and $K \cap \partial\Delta$ are nonempty. Then $\psi(a, \Delta) = \sup h(a, a)$, where the supremum is taken over all points $a \in \Delta \setminus G_a^0(\Delta)$.

Theorem 2. Let $f(x) \in [BL]_K^{n/2}$. Then the following lower estimate for the quantity K holds:

$$K \geq \sup_{0 < a < \delta} \min \left\{ \frac{\psi^n(a, \Delta)}{M_n n + \psi^n(a, \Delta)} \left[\ln \frac{R^n}{M_n} + \ln \ln \frac{\delta}{a} \right], \frac{R^n}{M_n} \ln \frac{\delta}{a} \right\}, \quad (1)$$

where M_n is an absolute constant (1).

The proof is based on the inequality of Theorem 2 from (1).

Corollary. If

$$\lim_{a \rightarrow 0} \psi(a, \Delta) \sqrt[n]{\ln \ln \frac{\delta}{a}} = \infty, \quad (2)$$

then there is no mapping $y = f(x)$ of class $[BL]^{n/2}$ of the ball D onto the domain Δ .

Remark. Let $y = f(x)$ be a Q -quasiconformal mapping of the ball $D : |x| < R$ onto the domain Δ of finite measure $m\Delta$, and let $f(0) = 0$. Then $f(x) \in [BL]_K^{n/2}$, where $K \leq n^{n/2} Q^{n-1} \max(mD, m\Delta)$, and hence

$$Q \geq [n^{n/2} \max(mD, m\Delta)]^{1/(n-1)} K^{1/(n-1)}. \quad (3)$$

Inequalities (1) and (3) give a lower estimate for the quasiconformality coefficient Q in terms of the structure of the domain Δ and its boundary, which is expressed by means of the function $\psi(a, \Delta)$.

The corollary of Theorem 2 makes it possible to construct various domains $\Delta \subset E^n$ ($n \geq 3$), homeomorphic to a ball, onto which the ball cannot be mapped by means of functions of class $[BL]^{n/2}$, nor by quasiconformal mappings. It also makes it possible to construct sequences of domains $\{\Delta_m\}$ such that there exist mappings of the ball $\{f_m\}$, $f_m(D) = \Delta_m$, $f_m \in [BL]^{n/2}$ ($m = 1, 2, \dots$), but the relation $f_m \in [BL]^K$ is not satisfied for any K .

Example 2. Consider in E^3 the cube

$$R = \{x \in E^3 : |x_i| < 1 \ (i = 1, 2, 3)\}.$$

Let $\{a_m\}$ be a sequence of positive numbers $0 < a_m < 1/3$ ($m = 1, 2, \dots$), monotonically tending to 0. Put

$$B_m = \{x \in E^3 : |x_1| < b_m, a_{2m-1} < x_2 < a_{2m}, 1 \leq x_3 < b_m\},$$

where

$$b_m = \{\ln[-\ln(a_{2m} - a_{2m-1})]\}^{-1/6}.$$

The set $\bar{\Delta}$,

$$\Delta = R \cup \left(\bigcup_{m=1}^{\infty} B_m \right),$$

is a domain homeomorphic to the closed ball; moreover, for this domain relation (2) holds, and hence there is no mapping of class $[BL]^{3/2}$ of the ball onto this domain.

Example 3. Consider the cube

$$R = \{x \in E^3 : |x_i| < 1 \ (i = 1, 2, 3)\}.$$

Let $\{a_m\}$ be a sequence of positive numbers $2/3 \leq a_m < 1$ ($m = 1, 2, \dots$), monotonically tending to 1. Put

$$C_m = \{x \in R : x_2 = a_m\}, \quad q_m = \{x \in C_m : |x_1| < a_{m+1} - a_m, |x_3| < b_m\},$$

where

$$b_m = \frac{1}{2} \exp[-\exp(a_{m+1} - a_m)^{-6}].$$

Then the set

$$\Delta = R \setminus \left[\bigcup_{m=1}^{\infty} (C_m \setminus q_m) \right]$$

is a domain, homeomorphic to a ball and compact in the relative distance (it is constructed according to the type of the domains of Example 1). The domain Δ has

a simple end of the second type, determined by the sequence of sections $\{q_m\}$, and for it relation (2) is satisfied.

Example 4. Consider a domain having the form of a wedge with zero angle $\Delta = \{x \in E^3 : |x_1| < 1, |x_2| < 1, |x_3| < \varphi(x_2)\}$, where $\varphi(x_2) = \exp[-\exp(x_2+1)^{-6}]$. It is easy to see that for this domain relation (2) is satisfied. A mapping of class $[BL]^{n/2}$ onto a domain of this form will be impossible also in the case when, as the function $\varphi(x_2)$, one takes one having a higher order of contact with the axis Ox_2 in comparison with the function chosen in this example.

We note that, for quasiconformal mappings, as F. W. Gehring and J. Väisälä established [11], a mapping onto a domain having a ridge directed outward, or a peak directed inward, is impossible. One can give examples of mappings

showing that, for mappings of class $[BL]^{n/2}$, such mappings are possible if the ridge and the peak are not too “sharp.”

Example 5. Put $\Delta = \{x \in E^3 : |x_i| < 1 \ (i = 1, 2, 3)\}$, $R_n = \{x \in \Delta : x_2 = 1 - 1/2n, -1 < x_3 \leq 0\}$ ($n = 1, 2, \dots$), and consider the sequence of domains $\{\Delta_n\}$, $\Delta_n = \Delta \setminus R_n$. Then there exists no number K and no sequence of mappings $\{f_n(x)\}$ of the ball D onto the domains Δ_n , $f_n(D) = \Delta_n$, such that $f_n(x) \in [BL]_K^{3/2}$ for every n .

Example 6. Put $\Delta = \{x \in E^3 : |x_1| < 1, |x_2| < 1, -1 < x_3 < 2\}$, $R'_n = \{x \in \Delta : x_2 \in [-1/2n, 1/2n], x_3 = 1\}$, $\Delta_n = \Delta \setminus R'_n$. Then there exists no sequence of mappings $\{f_n(x)\}$ of the ball D onto the domains Δ_n , $f_n(D) = \Delta_n$, and no number K such that $f_n \in [BL]_K^{3/2}$ for all n .

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Note: Figure translations are in progress. See original paper for figures.

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