

DUAL MODELS OF PRODUCTION

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Abstract

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MATHEMATICS

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DUAL MODELS OF PRODUCTION

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In the present note a positive-homogeneous regular model of production is considered. A dual model is defined and its properties are described. In the study of production models, equilibrium systems play an important role; with their help the asymptotic behavior of optimal trajectories of an economy is described. The simultaneous consideration of the original and the dual models makes it possible to clarify the structure of equilibrium systems.

1°. Let us consider finite-dimensional arithmetic normed spaces $X_{(1)}$ and $X_{(2)}$. By $X_{(i)}^*$ we denote the space conjugate to $X_{(i)}$, by $K_{(i)}$ (respectively $K_{(i)}^*$) the cone of vectors with nonnegative components of the space $X_{(i)}$ (respectively $X_{(i)}^*$). We shall assume that $X_{(i)}$ (respectively $X_{(i)}^*$) is partially ordered by means of the cone $K_{(i)}$ (respectively $K_{(i)}^*$).

For $\xi \in K_{(i)}$, $f \in K_{(i)}^*$, put

$$f(\xi) = \sup_{x \in \xi} f(x).$$

Similarly, if $\eta \in K_{(i)}^*$, $x \in K_{(i)}$, then

$$(\eta)x = \inf_{f \in \eta} f(x).$$

Consider a mapping a that assigns to each point $x \in K_{(1)}$ a subset $a(x)$ of the cone $K_{(2)}$. We shall assume henceforth that this mapping has the following properties:

- 1) $a(x)$ is bounded for every $x \in K_{(1)}$; (1)
- 2) if $x' \geq x''$, then $a(x') \supset a(x'')$; (2)
- 3) $a(x' + x'') \supset a(x') + a(x'')$; (3)
- 4) $a(\lambda x) = \lambda a(x)$ ($\lambda > 0$); (4)
- 5) if $x_n \rightarrow x$, $y_n \in a(x_n)$, $y_n \rightarrow y$, then $y \in a(x)$; (5)
- 6) $\bigcup_{x \in K_{(1)}} a(x) = K_{(2)}$. (6)

Let us introduce, alongside the mapping a , the conjugate mapping a^* , assigning to each $f \in K_{(2)}^*$ a subset $a^*(f)$ of the cone $K_{(1)}^*$, where $a^*(f)$ is defined by the formula

$$a^*(f) = \{g \in K_{(1)}^* \mid f(a(x)) \leq g(x) \text{ for every } x \in K_{(1)}\}.$$

Lemma 1. For every $f \in K_{(2)}^*$ the set $a^*(f)$ is nonempty, convex, closed, and has the following property: if $g \in a^*(f)$, $g' \geq g$, then $g' \in a^*(f)$; $a^*(f) = K_{(1)}^*$ if and only if $f = 0$.

Lemma 2. For any $x \in K_{(1)}$, $f \in K_{(2)}^*$,

$$f(a(x)) = (a^*(f))x.$$

Lemma 3. The mapping a^* has the following properties:

- 1) if $f' \geq f''$, then $a^*(f') \subset a^*(f'')$;
- 2) $a^*(f' + f'') \supset a^*(f') + a^*(f'')$;
- 3) $a^*(\lambda f) = \lambda a^*(f)$ ($\lambda > 0$);
- 4) if $f_n \rightarrow f$, $g_n \in a^*(f_n)$, $g_n \rightarrow g$, then $g \in a^*(f)$;
- 5)

$$\bigcup_{f \in K_{(2)}^*} a^*(f) = K_{(1)}^*.$$

Lemma 4.

$$\max_{\substack{x \in K_{(1)}; y \in a(x) \\ \|x\| \leq 1}} \|y\| = \max_{\substack{f \in K_{(2)}^*; g \in a^*(f) \\ \|f\| \leq 1}} \|g\|.$$

Introduce for consideration the second conjugate to a , the mapping a^{**} , which assigns to each $x \in K_{(1)}$ a subset $a^{**}(x)$ of the cone $K_{(2)}$, where

$$a^{**}(x) = \{y \in K_{(2)} \mid (a^*(f))x \leq f(y) \text{ for any } f \in K^{(2)}\}.$$

Lemma 5. An element $y \in a^{**}(x)$ if and only if there exists $z \in a(x)$ such that $y \leq z$.

It follows from Lemma 5, in particular, that if the mapping a is such that, for any $x \in K_{(1)}$, the set $a(x)$, together with each of its points y , contains the conic segment $\langle 0, y \rangle$, then $a^{**} = a$. Note that, whatever the mapping a satisfying conditions (1)–(6) may be, $a^{***} = a^*$.

2°. Consider a system $\{X_t\}_{t=1}^\infty$ of finite-dimensional arithmetic normed spaces; a system $\{\tilde{K}_t\}_{t=1}^\infty$, where $\tilde{K}_t \subset X_t$, $\tilde{K}_t$ is a cone with vertex at zero, containing no straight line; and a system $\{\tilde{a}_t\}_{t=1}^\infty$, where $\tilde{a}_t$ is a mapping assigning to each point $x \in \tilde{K}_t$ a certain subset $\tilde{a}_t(x)$ of the cone $\tilde{K}_{t+1}$. The production model $\mathfrak{M}$ is defined as the totality of the systems $\{X_t\}_{t=1}^\infty, \{\tilde{K}_t\}_{t=1}^\infty, \{\tilde{a}_t\}_{t=1}^\infty$. Thus, by definition,

$$\mathfrak{M} = (\{X_t\}_{t=1}^\infty; \{\tilde{K}_t\}_{t=1}^\infty; \{\tilde{a}_t\}_{t=1}^\infty).$$

As above, by X_t^* ($t = 1, 2, \dots$) we shall mean the space conjugate to X_t ; by K_t (respectively K_t^*) we shall mean the cone of vectors with nonnegative components in the space X_t (respectively X_t^*).

The production model $\mathfrak{M}$ is called a regular positively homogeneous model if

$$\mathfrak{M} = (\{X_t\}_{t=1}^\infty; \{K_t\}_{t=1}^\infty; \{a_t\}_{t=1}^\infty),$$

where a_t is a mapping possessing properties 1)–6).

In what follows, by the symbol $\mathfrak{M}$ we shall always denote a regular positively homogeneous model.

Consider the model

$$\mathfrak{M} = (\{X_t\}_{t=1}^\infty; \{K_t\}_{t=1}^\infty; \{a_t\}_{t=1}^\infty).$$

The model $\mathfrak{M}^*$ dual to $\mathfrak{M}$ is defined as follows:

$$\mathfrak{M}^* = (\{X_t^*\}_{t=1}^\infty; \{K_t^*\}_{t=1}^\infty; \{a_t^*\}_{t=1}^\infty).$$

The second dual model $\mathfrak{M}^{**}$ to $\mathfrak{M}$ is defined as follows:

$$\mathfrak{M}^{**} = (\{X_t\}_{t=1}^{\infty}; \{K_t\}_{t=1}^{\infty}; \{a_t^{**}\}_{t=1}^{\infty}).$$

If the mappings a_t ($t = 1, 2, \dots$) are such that from the conditions $x \in K_t$, $y \in a_t(x)$, $y' \leq y$ it follows that $y' \in a_t(x)$ (the model $\mathfrak{M}$ admits “free-destruction of products”), then, as is easy to see, $\mathfrak{M} = \mathfrak{M}^{**}$. (In the general case $\mathfrak{M}^{**}$ is, in the terminology of Nikaido [2], the disposal hull of the model $\mathfrak{M}$.)

Let us introduce the following definitions. An infinite-dimensional trajectory of the model $\mathfrak{M}$ will mean a sequence $\{x_t\}_{t=1}^{\infty}$ such that $x_1 \in K_1$, $x_t \in a_{t-1}(x_{t-1})$ ($t = 2, 3, \dots$). We shall say that the infinite trajectory indicated above starts from $x = x_1$. An infinite trajectory of the model $\mathfrak{M}^*$ will mean a sequence $\{f_t\}_{t=\infty}^1$ such that $f_t \in a_t^*(f_{t+1})$ ($t = \dots, 3, 2, 1$). We shall say that the trajectory indicated above arrives at $f = f_1$.

For $x \in K_1$ put

$$A_0(x) = x, \quad A_t(x) = \bigcup_{y \in A_{t-1}(x)} a_t(y) \quad (t = 1, 2, \dots).$$

For $f \in K_t^*$ put

$$A_0^*(f) = f, \quad A_t^*(f) = \bigcup_{g \in A_{t-1}^*(f)} a_{T-t}(g) \quad (t = 1, 2, \dots, T-1).$$

An infinite trajectory $\{x_t\}_{t=1}^{\infty}$ of the model $\mathfrak{M}$ will be called optimal if for every natural T there exists a functional $f \in K_T^*$ ($f \neq 0$) such that $f(x_T) = f(A_{T-1}(x))$. An infinite trajectory $\{f_t\}_{t=\infty}^1$ of the model $\mathfrak{M}^*$ will be called optimal if for every natural T there exists $x \in K_1$ ($x \neq 0$) such that $f_1(x) = (A_{T-1}^*(f_T))x$.

In the study of the production model $\mathfrak{M}$, equilibrium systems play an important role.

A pair of trajectories $(\bar{\chi}, \bar{\varphi})$ ($\bar{\chi} = \{\bar{x}_t\}_{t=1}^{\infty}$, $\bar{\varphi} = \{\bar{f}_t\}_{t=\infty}^1$) will be called an equilibrium system of the pair of models $(\mathfrak{M}, \mathfrak{M}^*)$, if

$$0 < \bar{f}_1(\bar{x}_1) = \bar{f}_2(\bar{x}_2) = \dots = \bar{f}_t(\bar{x}_t) = \dots$$

Let $(\bar{\chi}, \bar{\varphi})$ be an equilibrium system. Put $\alpha_t = \|\bar{f}_{t+1}\|/\|\bar{f}_t\|$, $\bar{f}'_t = \bar{f}_t/\|\bar{f}_t\|$. Then the following relations hold:

$$1) \quad \bar{f}'_1(\bar{x}_1) > 0; \quad (7)$$

$$2) \quad \bar{f}'_{t+1}(a_t(x)) \leq \alpha_t \bar{f}'_t(x) \quad (x \in K_t, t = 1, 2, \dots); \quad (8)$$

$$3) \quad \bar{f}'_{t+1}(\bar{x}_{t+1}) = \alpha_t \bar{f}'_t(\bar{x}_t) \quad (t = 1, 2, \dots). \quad (9)$$

Put $\bar{\varphi}' = \{\bar{f}'_t\}_{t=1}^\infty$, $\alpha = \{\alpha_t\}_{t=1}^\infty$. It follows from (7)–(9) that the equilibrium system $(\bar{\chi}, \bar{\varphi})$ of the pair of models $(\mathfrak{M}, \mathfrak{M}^*)$ generates an equilibrium system $(\bar{\chi}, \bar{\varphi}', \alpha)$ on K_1 of the model $\mathfrak{M}$ (see (1)). Conversely, every equilibrium system of the model $\mathfrak{M}$ generates an equilibrium system of the pair of models $(\mathfrak{M}, \mathfrak{M}^*)$.

A triple of sequences $(\bar{\chi}, \bar{\varphi}, \beta)$ ($\bar{\chi} = \{\bar{x}_t\}_{t=1}^\infty$, $\bar{x}_t \in K_t$, $\|\bar{x}_t\| = 1$, $\bar{\varphi} = \{\bar{f}_t\}_{t=1}^\infty$ is an infinite trajectory of the model $\mathfrak{M}^*$; $\beta = \{\beta_t\}_{t=1}^\infty$, β_t is a positive number) will be called an equilibrium system of the model $\mathfrak{M}^*$, if:

- 1) $\bar{f}_1(\bar{x}_1) > 0$;
- 2) $(a_t^* f) \bar{x}_t \geq \beta_t f(\bar{x}_{t+1}) \quad (f \in K_{t+1}^*, t = \dots, 3, 2, 1)$;
- 3) $\bar{f}_t(\bar{x}_t) = \beta_t \bar{f}_{t+1}(\bar{x}_{t+1}) \quad (t = \dots, 3, 2, 1)$.

It is easy to verify that every equilibrium system $(\bar{\chi}, \bar{\varphi})$ of the pair $(\mathfrak{M}, \mathfrak{M}^*)$ ($\bar{\chi} = \{\bar{x}_t\}_{t=1}^\infty$, $\bar{\varphi} = \{\bar{f}_t\}_{t=1}^\infty$) generates an equilibrium system $(\bar{\chi}', \bar{\varphi}, \beta)$ of the model $\mathfrak{M}^*$. (Here $\bar{\chi}' = \{\bar{x}_t / \|\bar{x}_t\|\}_{t=1}^\infty$, $\beta = \{\|\bar{x}_{t+1}\| / \|\bar{x}_t\|\}_{t=1}^\infty$.) If $\mathfrak{M} = \mathfrak{M}^{**}$, then the converse is also true: every equilibrium system of the model $\mathfrak{M}^*$ generates an equilibrium system of the pair $(\mathfrak{M}, \mathfrak{M}^*)$.

The construction of equilibrium systems is described by the following theorems.

Theorem 1. Let $(\bar{\chi}, \bar{\varphi})$ be an equilibrium system of the pair $(\mathfrak{M}, \mathfrak{M}^*)$. Then $\bar{\chi}$ is an infinite optimal trajectory of the model $\mathfrak{M}$, and $\bar{\varphi}$ is an infinite optimal trajectory of the model $\mathfrak{M}^*$.

Theorem 2. Whatever the point $x \in K_1$, there exists an infinite optimal trajectory of the model $\mathfrak{M}$ issuing from x ; whatever the interior point f of the cone K_1^* , there exists an infinite optimal trajectory of the model $\mathfrak{M}^*$ arriving at f .

Theorem 3 (see (1)). Let x be an interior point of K_1 . Then, whatever infinite optimal trajectory χ of the model $\mathfrak{M}$ issuing from x is given, there exists an infinite trajectory φ of the model $\mathfrak{M}^*$ such that the pair (χ, φ) forms an equilibrium system $(\mathfrak{M}, \mathfrak{M}^*)$.

Theorem 4. Let $\mathfrak{M} = \mathfrak{M}^{**}$, and let f be an interior point of K_1^* . Then, whatever infinite optimal trajectory φ of the model $\mathfrak{M}^*$ arriving at f is given, there exists an infinite trajectory χ of the model $\mathfrak{M}$ such that (χ, φ) forms an equilibrium system of the pair $(\mathfrak{M}, \mathfrak{M}^*)$.

Let us consider separately the Gale model $\mathfrak{M}_G$ (see (3)), which is a special case of the model considered by us. In this case $X_1 = X_2 = \dots = X_t = \dots = X$;

$K_1 = K_2 = \dots = K_t = \dots = K$; $a_1 = a_2 = \dots = a_t = \dots = a$. Allowing a liberty of language, one may say that $\mathfrak{M}_G = (X, K, a)$. We shall assume that the mapping a , which defines the model $\mathfrak{M}_G$, satisfies the condition of "free disposal of products," i.e. $\mathfrak{M}_G = \mathfrak{M}_G^{**}$. An equilibrium state of the model $\mathfrak{M}_G$, as is known, is a triple $(\bar{x}, \bar{f}, \alpha)$ ($\bar{x} \in K, \bar{f} \in K^*, \alpha$ is a positive number) such that:

- 1) $\bar{f}(\bar{x}) > 0$;
- 2) $\bar{f}(a(x)) \geq \alpha \bar{f}(x)$ ($x \in K$);
- 3) $\alpha \bar{x} \in a(\bar{x})$.

An equilibrium state of the model $\mathfrak{M}_G^*$ will be called a triple $(\bar{x}, \bar{f}, \alpha)$ ($\bar{x} \in K, \bar{f} \in K^*, \alpha$ is a positive number) such that:

- 1) $\bar{f}(\bar{x}) > 0$;
- 2) $\alpha \bar{f}(\bar{x}) \geq (a^*(\bar{f}))\bar{x}$ ($\bar{f} \in K^*$);
- 3) $\alpha \bar{f} \in a^*(\bar{f})$.

From the definitions it follows immediately:

Theorem 5. In order that a triple $(\bar{x}, \bar{f}, \alpha)$ be an equilibrium state of the model $\mathfrak{M}_G^*$, it is necessary and sufficient that it be an equilibrium state of the model $\mathfrak{M}_G$.

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2. H. Nikaido, *Econometrica*, **32**, Nos. 1-2 (1964).
3. D. Gale, in: *Linear Inequalities and Related Questions*, Moscow, 1959.

Note: Figure translations are in progress. See original paper for figures.

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