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Abstract

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ASTRONOMY

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DIFFUSION OF RADIATION IN A MEDIUM OF LARGE OPTICAL THICKNESS WITH ANISOTROPIC SCATTERING

In the study of planetary atmospheres and water basins one encounters the problem of the diffusion of radiation in a plane layer of large optical thickness illuminated by parallel rays. For the case of a spherical scattering indicatrix this problem was solved earlier ⁽¹⁾. Now, by the same method, it is solved for the case of anisotropic scattering of light. In the first part of the present note the radiation field in the deep layers of a semi-infinite medium is determined; in the second, asymptotic formulas are derived for the intensity of radiation emerging from a plane layer of large optical thickness.

1. Let parallel rays, making an angle $\arccos \zeta$ with the normal, fall on a semi-infinite medium and produce an illumination of the area perpendicular to them equal to πS . Denote by $I(\tau, \eta, \zeta)$ the azimuth-averaged intensity of the diffuse radiation traveling at optical depth τ at an angle $\arccos \eta$ to the normal, and by $B(\tau, \eta, \zeta)$ the azimuth-averaged "source function." As is known, these quantities are related by the equations:

$$\eta dI(\tau, \eta, \zeta)/d\tau = -I(\tau, \eta, \zeta) + B(\tau, \eta, \zeta), \quad (1)$$

$$B(\tau, \eta, \zeta) = \frac{\lambda}{2} \int_{-1}^{+1} p(\eta, \eta') I(\tau, \eta', \zeta) d\eta' + \frac{\lambda}{4} Sp(\eta, \zeta) e^{-\tau/\zeta}; \quad (2)$$

where λ is the photon survival probability in an elementary act of scattering,

$$p(\eta, \eta') = \frac{1}{2\pi} \int_0^{2\pi} \chi(\eta\eta' + \sqrt{(1-\eta^2)(1-\eta'^2)} \cos \varphi) d\varphi, \quad (3)$$

$\chi(\cos \gamma)$ is the scattering indicatrix.

From (1) and (2), with the boundary condition $I(0, \eta, \zeta) = 0$ for $\eta > 0$, we obtain an integral equation for determining the source function:

$$\begin{aligned}
 B(\tau, \eta, \zeta) = & \frac{\lambda}{2} \int_0^1 p(\eta, \eta') d\eta' \int_0^\tau B(\tau', \eta', \zeta) e^{-(\tau-\tau')/\eta'} \frac{d\tau'}{\eta'} \\
 & + \frac{\lambda}{2} \int_0^1 p(\eta, -\eta') d\eta' \int_\tau^\infty B(\tau', -\eta', \zeta) e^{-(\tau'-\tau)/\eta'} \frac{d\tau'}{\eta'} \quad (4) \\
 & + \frac{\lambda S}{4} p(\eta, \zeta) e^{-\tau/\zeta}.
 \end{aligned}$$

For the deep layers of a semi-infinite medium we have

$$B(\tau, \eta, \zeta) = Sc(\zeta)b(\eta)e^{-k\tau}, \quad (5)$$

$$I(\tau, \eta, \zeta) = Sc(\zeta)i(\eta)e^{-k\tau}, \quad (6)$$

where

$$i(\eta) = b(\eta)/(1 - k\eta). \quad (7)$$

In this case equation (4) can be written in the form

$$B(\tau, \eta, \xi) = \frac{\lambda}{2} \int_0^1 p(\eta, \eta') d\eta' \int_{-\infty}^\tau B(\tau', \eta', \xi) e^{-(\tau-\tau')/\eta'} \frac{d\tau'}{\eta'} + \frac{\lambda}{2} \int_0^1 p(\eta, -\eta') d\eta' \int_\tau^\infty B(\tau', -\eta', \xi) e^{-(\tau'-\tau)/\eta'} \frac{d\tau'}{\eta'} \quad (8)$$

Substituting (5) into (8), we arrive at an equation for determining the function $b(\eta)$

$$b(\eta) = \frac{\lambda}{2} \int_{-1}^{+1} p(\eta, \eta') \frac{b(\eta')}{1 - k\eta'} d\eta'. \quad (9)$$

The quantity k is found from the solvability condition for this equation. For normalizing $b(\eta)$ one may use the condition

$$\frac{1}{2} \int_{-1}^{+1} b(\eta) d\eta = 1. \quad (10)$$

Equation (9) was first obtained by V. A. Ambartsumian ⁽²⁾. It was subsequently solved in works by the author ⁽³⁾, Kuper ⁽⁴⁾, and others. However, as far as we know, the function $c(\xi)$ has not yet been determined. To determine it, let us differentiate both sides of equation (4) with respect to τ and compare the result obtained with equation (4). Doing this, we find

$$B'(\tau, \eta, \xi) = -\frac{1}{\xi} B(\tau, \eta, \xi) + \frac{2}{S} \int_0^1 B(\tau, \eta, \eta') B(0, \eta', \xi) \frac{d\eta'}{\eta'}. \quad (11)$$

Substitution of (5) into (11) gives

$$c(\xi) = \frac{\xi}{1 - k\xi} \frac{2}{S} \int_0^1 c(\eta) B(0, \eta, \xi) \frac{d\eta}{\eta}. \quad (12)$$

The quantity $B(0, \eta, \xi)$ entering (12) can be found from equation (4). Setting $\tau = 0$ in it, we obtain

$$B(0, \eta, \xi) = \frac{\lambda}{4} S \left[2\xi \int_0^1 p(\eta, -\eta') \rho(\eta', \xi) d\eta' + p(\eta, \xi) \right], \quad (13)$$

where $\rho(\eta, \xi)$ denotes the brightness coefficient of a semi-infinite medium, defined by the formula

$$\int_0^\infty B(\tau, -\eta, \xi) e^{-\tau/\eta} \frac{d\tau}{\eta} = S \rho(\eta, \xi) \xi. \quad (14)$$

Comparing equations (12) and (13) with the equations obtained in the work of V. A. Ambartsumian ⁽⁵⁾, we see that the function $c(\xi)$ can be represented in the form

$$c(\xi) = u(\xi) \xi, \quad (15)$$

where $u(\eta)$ is the intensity of radiation diffusely transmitted by a semi-infinite medium at an angle $\arccos \eta$ to the normal. Previously the function $u(\eta)$ was determined only up to a constant factor. We shall now find this factor for the case of deep layers of a semi-infinite medium. To do this, we apply the following device. Substitute in (8), instead of

Replace, in $B(\tau, \eta, \xi)$, the function by $b(\eta) e^{-\kappa\tau}$ and write the identity obtained in the form

$$\begin{aligned} b(\eta) e^{-\kappa\tau} &= \frac{\lambda}{2} \int_0^1 p(\eta, \eta') d\eta' \int_0^\tau b(\eta') e^{-\kappa\tau' - (\tau - \tau')/\eta'} \frac{d\tau'}{\eta'} \\ &+ \frac{\lambda}{2} \int_0^1 p(\eta, -\eta') d\eta' \int_\tau^\infty b(-\eta') e^{-\kappa\tau' - (\tau' - \tau)/\eta'} \frac{d\tau'}{\eta'} \\ &+ \frac{\lambda}{2} \int_0^1 p(\eta, \eta') e^{-\tau/\eta'} i(\eta') d\eta'. \end{aligned} \quad (16)$$

Comparing (16) with (4), we have

$$b(\eta)e^{-\kappa\tau} = \frac{2}{S} \int_0^1 B(\tau, \eta, \eta') i(\eta') d\eta'. \quad (17)$$

Relation (17) is valid for arbitrary τ . Substituting expression (5) into (17) and using formula (15), we obtain

$$2 \int_0^1 u(\eta) i(\eta) \eta d\eta = 1. \quad (18)$$

Formula (18) also makes it possible to find the unknown constant factor in the function $u(\eta)$. Thus the problem of determining the radiation field in deep layers of a semi-infinite medium is completely solved.

2. Let us now consider the diffusion of radiation in a plane layer of large optical thickness τ_0 . As before, suppose that the layer is illuminated by parallel rays.

We shall be interested in the intensities of the radiation emerging from the layer, or in the corresponding brightness coefficients $\rho(\eta, \xi, \tau_0)$ and $\sigma(\eta, \xi, \tau_0)$, related to the intensities by the relations

$$I(0, -\eta, \xi, \tau_0) = S\rho(\eta, \xi, \tau_0)\xi, \quad I(\tau_0, \eta, \xi, \tau_0) = S\sigma(\eta, \xi, \tau_0)\xi. \quad (19)$$

In the case under consideration (for $\tau_0 \gg 1$), the results obtained above for deep layers of a semi-infinite medium can be used to determine the brightness coefficients. Let us make, mentally, a cut in the semi-infinite medium at a large optical depth τ_0 . Then it may be assumed that a plane layer of optical thickness τ_0 is illuminated from above by parallel rays, and from below by diffuse radiation, whose intensity is determined by formula (6) at $\tau = \tau_0$ and $\eta < 0$.

On the basis of what has been said, we have the following relations:

$$\rho(\eta, \xi) = \rho(\eta, \xi, \tau_0) + 2u(\xi)e^{-\kappa\tau_0} \int_0^1 \sigma(\eta, \eta', \tau_0) i(-\eta') \eta' d\eta', \quad (20)$$

$$u(\xi) i(\eta) e^{-\kappa\tau_0} = \sigma(\eta, \xi, \tau_0) + 2u(\xi) e^{-\kappa\tau_0} \int_0^1 \rho(\eta, \eta', \tau_0) i(-\eta') \eta' d\eta'. \quad (21)$$

Here, as above, $\rho(\eta, \xi)$ denotes the brightness coefficient of the semi-infinite medium, assumed known.

From equations (20) and (21) we easily obtain

$$\rho(\eta, \xi, \tau_0) = \rho(\eta, \xi) - \frac{u(\eta)u(\xi)}{1 - N^2 e^{-2\kappa\tau_0}} M N e^{-2\kappa\tau_0}, \quad (22)$$

$$\sigma(\eta, \xi, \tau_0) = \frac{u(\eta)u(\xi)}{1 - N^2 e^{-2\kappa\tau_0}} M e^{-\kappa\tau_0}, \quad (23)$$

where

$$M u(\eta) = i(\eta) - 2 \int_0^1 \rho(\eta, \eta') i(-\eta') \eta' d\eta', \quad (24)$$

$$N = 2 \int_0^1 u(\eta) i(-\eta) \eta d\eta. \quad (25)$$

To determine the constant M , we use formula (18), as well as the relation

$$i(-\eta) = 2 \int_0^1 \rho(\eta, \eta') i(\eta') \eta' d\eta', \quad (26)$$

which follows from (17) and (14). Multiplying (24) by $i(\eta)\eta$ and integrating with respect to η from 0 to 1, with the aid of the indicated formulas we find

$$M = 2 \int_{-1}^1 i^2(\eta) \eta d\eta. \quad (27)$$

Thus, for the desired brightness coefficients $\rho(\eta, \xi, \tau_0)$ and $\sigma(\eta, \xi, \tau_0)$ we have obtained the asymptotic formulas (22) and (23), in which the constants M and N are determined by formulas (25) and (27), and the function $u(\eta)$ is normalized according to (18).

To determine the brightness coefficients in the case of pure scattering (i.e., for $\lambda = 1$), in formulas (22) and (23) one must set $k \rightarrow 0$. For small k we have

$$i(\eta) = 1 + \frac{3}{3 - x_1} k \eta, \quad (28)$$

where x_1 is the first coefficient in the expansion of the scattering indicatrix in Legendre polynomials. Substituting (28) into (27), we obtain

$$M = 8k/(3 - x_1), \quad (29)$$

and from formulas (25) and (18) it follows that

$$N = 1 - \frac{12k}{3 - x_1} \int_0^1 u(\eta) \eta^2 d\eta, \quad (30)$$

where the function $u(\eta)$ now refers to the case $\lambda = 1$.

Substitution of (29) and (30) into (22) and (23) for $k = 0$ gives

$$\rho(\eta, \xi, \tau_0) = \rho(\eta, \xi) - {}^4/{}_3 u(\eta) u(\xi) / [(1 - x_1/3) \tau_0 + \delta], \quad (31)$$

$$\sigma(\eta, \xi, \tau_0) = {}^4/{}_3 u(\eta) u(\xi) / [(1 - x_1/3) \tau_0 + \delta], \quad (32)$$

where

$$\delta = 2 \int_0^1 u(\eta) \eta^2 d\eta / \int_0^1 u(\eta) \eta d\eta. \quad (33)$$

Recently, in an article by van de Hulst and Grossman (6), formulas analogous to (22) and (23) were presented, but obtained by another method. They are a generalization of formulas previously obtained by the author (1, 7) for the case of isotropic scattering.

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CITED LITERATURE

1. V. V. Sobolev, *Transfer of Radiant Energy in the Atmospheres of Stars and Planets*, Moscow, 1956.
2. V. A. Ambartsumian, *Izv. AN SSSR, ser. geogr. i geofiz.*, **3**, 97 (1942).
3. V. V. Sobolev, *Izv. AN SSSR, ser. geogr. i geofiz.*, **8**, 273 (1944).
4. I. Kuscer, *J. Math. and Phys.*, **34**, 256 (1956).
5. V. A. Ambartsumian, *DAN*, **43**, 106 (1944).
6. H. C. van de Hulst, K. Grossman, *Proc. of Conference on the Atmospheres of Mars and Venus*, Tucson, 1967.
7. V. V. Sobolev, *DAN*, **155**, 316 (1964).

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