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Abstract**Full Text**

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MATHEMATICS

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ON THE APPROXIMATION OF CONTINUOUS PERIODIC FUNCTIONS BY LINEAR OPERATORS CONSTRUCTED ON THE BASIS OF THEIR FOURIER SERIES*(Presented by Academician S. N. Bernstein on 28 XII 1967)*

Let $f(x)$ be a continuous periodic function of period 2π with Fourier series

$$\sum_{n=0}^{\infty} A_n(x),$$

where $A_0(x) = a_0/2$, $A_n(x) = a_n \cos nx + b_n \sin nx$ ($n = 1, 2, \dots$). With the aid of an arbitrary triangular matrix of numbers $\{\lambda_\nu(n)\}$ ($\lambda_0(n) = 1$, $\lambda_\nu(n) = 0$ for $\nu > n$), we construct the sequence of operators

$$U_n(f; x; \lambda) = \sum_{\nu=0}^n \lambda_\nu(n) A_\nu(x) \quad (n = 0, 1, 2, \dots). \quad (1)$$

Along with operators of the form (1), let us introduce for consideration the operators

$$U_r^*(f; x; \gamma) = \sum_{\nu=0}^{\infty} \gamma_\nu(r) S_\nu(f; x), \quad (2)$$

where

$$S_\nu(f; x) = \sum_{k=0}^{\nu} A_k(x),$$

and the system of functions $\{\gamma_\nu(r)\}$ is defined on some set E of the real axis and has the following properties: 1)

$$\sum_{\nu=0}^{\infty} \gamma_\nu(r) = 1$$

for every $r \in E$; 2) the series (2) converges uniformly for every $r \in E$.

Below we establish the following two assertions, giving estimates of the deviation of the function $f(x)$ from operators of the form (1) or (2) in terms of the sequence of its best approximations:

$$E_n(f) = \inf_{T_n} \max_x |f(x) - T_n(x)|.$$

Theorem 1. *For every continuous function $f(x)$ of period 2π , the inequality*

$$|f(x) - U_n(f; x; \lambda)| \leq (1 + L_n)E_n(f) + 2 \sum_{\nu=0}^{m-1} \rho(2^{\nu+1}; n)E_{2^{\nu}-1}(f) + 2\rho(n; n)E_{2^m}(f), \quad (3)$$

holds, where

$$2^m < n \leq 2^{m+1}, \quad L_n = \frac{2}{\pi} \int_0^\pi \left| \frac{1}{2} + \sum_{\nu=1}^n \lambda_\nu(n) \cos \nu x \right| dx, \quad \rho(\mu; n) = \frac{1}{\mu} \sum_{k=0}^{2\mu-1} |F_{k,\mu}(\lambda)|, \quad F_{k,\mu}(\lambda) = \frac{1 - \lambda_\mu(n)}{2} + \sum_{\nu=0}^{\mu-1} (1 - \lambda_{\mu-\nu}(n)) \cos \frac{\nu k \pi}{n}.$$

Theorem 2. *For the deviation of a continuous function $f(x)$ of period 2π from operators of the form (2), for every $r \in E$ the estimate*

$$|f(x) - U_r^*(f; x; \gamma)| \leq 2\gamma_0(r)E_0(f) + 12 \sum_{\nu=0}^{\infty} E_{2^\nu}(f)\delta(\nu; r), \quad (4)$$

holds, where

$$\delta(\nu; r) = \left\{ 2^\nu \sum_{k=2^\nu}^{2^{\nu+1}-1} \gamma_k^2(r) \right\}^{1/2}.$$

We note that from the general inequalities (3) and (4) one can easily obtain, as consequences, the known (see ⁽¹⁻³⁾) estimates for the deviation of continuous functions of period 2π from Fejér sums, Abel–Poisson sums, Zygmund means, etc. In particular, for Fejér sums, inequality (3) implies the estimate

$$\left| f(x) - \frac{1}{n+1} \sum_{\nu=0}^n S_\nu(f; x) \right| \leq \frac{10}{n+1} \sum_{\nu=0}^n E_\nu(f). \quad (5)$$

The method applied by S. B. Stechkin ⁽¹⁾, using estimates of deviation for Vallée-Poussin sums, enabled him in this case to obtain the inequality

$$\left| f(x) - \frac{1}{n+1} \sum_{\nu=0}^n S_\nu(f; x) \right| \leq \frac{12}{n+1} \sum_{\nu=0}^n E_\nu(f).$$

Proof of Theorem 1. Let

$$T_\mu(x) = \sum_{\nu=0}^{\mu} C_\nu(x) = \sum_{\nu=0}^{\mu} \alpha_\nu \cos \nu x + \beta_\nu \sin \nu x$$

be an arbitrary trigonometric polynomial, and

$$U(T_\mu; x; \lambda) = \sum_{\nu=0}^{\mu} \lambda_\nu(n) C_\nu(x) \quad (\mu \leq n).$$

The identity holds

$$U(T_\mu; x; \lambda) = T_\mu(x) - \frac{1}{\mu} \sum_{k=0}^{2\mu-1} (-1)^k T_\mu \left(x + \frac{k\pi}{\mu} \right) F_{k,\mu}(\lambda), \quad (6)$$

where the numbers $F_{k,\mu}(\lambda)$ are defined in Theorem 1. Identity (6) is verified directly for the functions $\sin \nu x$, $\cos \nu x$ ($\nu = 0, 1, \dots, \mu$), after which its validity for the polynomial $T_\mu(x)$ follows. Now, choosing a sequence of polynomials $\{T_n(x)\}$ which, for each n , realize the best uniform approximation of order n to the function $f(x)$, we introduce into consideration the functions

$$R(\mu; \nu; n; x) = \frac{1}{\mu} \sum_{k=0}^{2\mu-1} (-1)^k T_\mu \left(x + \frac{k\pi}{\mu} \right) F_{k,\nu}(\lambda),$$

where $1 \leq \mu \leq \nu \leq n$. It is not difficult to verify that these functions satisfy the following relations: for $\nu > \mu$, $R(\mu; \nu; n; x) = R(\mu; \mu; n; x)$, and $R(n; n; n; x) = T_n(x) - U(T_n; x; \lambda)$. The estimate is known (see ⁽³⁾, p. 591)

$$|f(x) - U_n(f; x; \lambda)| \leq (1 + L_n) E_n(f) + \max_x |T_n(x) - U(T_n; x; \lambda)|. \quad (7)$$

Using the above-indicated properties of the functions $R(\mu; \nu; n; x)$, we find that for $2^m < n \leq 2^{m+1}$,

$$\begin{aligned} |T_n(x) - U(T_n; x; \lambda)| &\leq |R(2; 2; n; x) - R(0; 2; n; x)| + |R(n; n; n; x) - R(2^m; n; n; x)| \\ &\quad + \sum_{\nu=1}^{m-1} |R(2^{\nu+1}; 2^{\nu+1}; n; x) - R(2^\nu; 2^{\nu+1}; n; x)| \end{aligned}$$

$$(R(0; \nu; n; x) = T_0(x) - U(T_0; x; \lambda)). \quad (8)$$

Applying identity (6) to the polynomial $P_{2^{\nu+1}}(x) = T_{2^{\nu+1}}(x) - T_{2^\nu}(x)$ ($0 \leq \nu \leq m-1$), we obtain

$$\begin{aligned} R(2^{\nu+1}; 2^{\nu+1}; n; x) - R(2^\nu; 2^{\nu+1}; n; x) = \\ = \frac{1}{2^{\nu+1}} \sum_{k=0}^{2^{\nu+2}-1} (-1)^k P_{2^{\nu+1}} \left(x + \frac{k\pi}{2^{\nu+2}} \right) F_{k, 2^{\nu+1}}(\lambda). \end{aligned}$$

Hence it follows that

$$|R(2^{\nu+1}, 2^{\nu+1}, n; x) - R(2^\nu, 2^{\nu+1}, n; x)| \leq 2E_{2^\nu}(f)\rho(2^{\nu+1}, n). \quad (9)$$

In view of (8) and (9), we find that

$$|T_n(x) - U(T_n; x; \lambda)| \leq 2 \sum_{\nu=0}^{m-1} E_{2^{\nu-1}}(f)\rho(2^{\nu+1}, n) + E_{2^m}(f)\rho(n; n). \quad (10)$$

From the estimates (7) and (10) inequality (3) follows.

Theorem 2 is a consequence of the following stronger assertion.

Theorem 3. Let $f(x)$ be a continuous function of period 2π , and let $\{\gamma_\nu(r)\}$ be an arbitrary sequence of functions ($\nu = 0, 1, 2, \dots$) defined on some set E of the real axis. Then from the convergence of the series

$$\sum_{\nu=0}^{\infty} E_{2^\nu}(f)\delta(\nu; r) \quad (r \in E)$$

it follows that, for any x , the series

$$\sum_{\nu=0}^{\infty} |\gamma_\nu(r)| |f(x) - S_\nu(f; x)|$$

converges and, moreover, the estimate

$$\sum_{\nu=0}^{\infty} |\gamma_\nu(r)| |f(x) - S_\nu(f; x)| \leq 2E_0(f)\gamma_0(r) + 12 \sum_{\nu=0}^{\infty} E_{2^\nu}(f)\delta(\nu; r), \quad (11)$$

holds, where $\delta(\nu; r)$ are the numbers defined in Theorem 2.

Proof. Let $\{T_n(x)\}$ be a sequence of trigonometric polynomials which, for each n ($n = 0, 1, 2, \dots$), give the best approximation of order n to the function $f(x)$ in the uniform metric. Obviously, for any $m = 0, 1, 2, \dots$

$$\begin{aligned}\theta_m(f; x; r) &= \sum_{\nu=2^m}^{2^{m+1}-1} |\gamma_\nu(r)| |f(x) - S_\nu(f; x)| = \\ &= \sum_{\nu=2^m}^{2^{m+1}-1} |\gamma_\nu(r)| \left| \frac{1}{\pi} \int_0^\pi \varphi_m(x; t) \frac{\sin(\nu + 1/2)t}{2 \sin t/2} dt \right|,\end{aligned}$$

where

$$\varphi_m(x; t) = f(x+t) - T_{2^m}(x+t) - 2[f(x) - T_{2^m}(x)] + f(x-t) - T_{2^m}(x-t).$$

Since $|\varphi_m(x; t)| \leq 4E_{2^m}(f)$, we find

$$I_1 = \left| \frac{1}{\pi} \int_0^{\pi/2^m} \varphi_m(x; t) \frac{\sin(\nu + 1/2)t}{2 \sin t/2} dt \right| \leq 2(2\nu + 1) \frac{1}{2^m} E_{2^m}(f). \quad (12)$$

Let us now estimate the integral

$$\begin{aligned}I_2 &= \left| \frac{1}{\pi} \int_{\pi/2^m}^\pi \frac{\varphi_m(x; t)}{2 \operatorname{tg} t/2} \sin \nu t dt + \frac{1}{2\pi} \int_{\pi/2^m}^\pi \varphi_m(x; t) \cos \nu t dt \right| \leq \\ &\leq 2E_{2^m}(f) + \left| \frac{1}{\pi} \int_{\pi/2^m}^\pi \frac{\varphi_m(x; t)}{2 \operatorname{tg} t/2} \sin \nu t dt \right|.\end{aligned} \quad (13)$$

Fixing x , consider the function

$$\psi_m(x; t) = \begin{cases} \frac{\varphi_m(x; t)}{4 \operatorname{tg} t/2}, & \frac{\pi}{2^m} \leq t \leq \pi, \\ 0, & 0 \leq t < \frac{\pi}{2^m}, \end{cases}$$

$$\psi_m(x; -t) = -\psi_m(x; t), \quad \psi_m(x; t + 2\pi) = \psi_m(x; t).$$

The second term on the right-hand side of inequality (13), for any $\nu = 1, 2, \dots$, is a Fourier coefficient of the function $\psi_m(x; t)$. Therefore, by Bessel's inequality,

$$\left\{ \sum_{\nu=2^m}^{2^{m+1}-1} \left| \frac{1}{\pi} \int_{\pi/2^m}^{\pi} \frac{\varphi_m(x; t)}{2 \operatorname{tg} t/2} \sin \nu t dt \right|^2 \right\}^{1/2} \leq \left\{ \frac{2}{\pi} \int_0^{\pi} |\psi_m(x; t)|^2 dt \right\}^{1/2} \leq \frac{2}{\sqrt{\pi}} 2^{m/2} E_{2^m}(f). \quad (14)$$

Using estimates (12), (13), (14) and applying Bunyakovsky's inequality, we obtain

$$\begin{aligned} \theta_m(f; x; r) &\leq \frac{2E_{2^m}(f)}{2^m} \sum_{\nu=2^m}^{2^{m+1}-1} |\gamma_{\nu}(r)|(2\nu+1) + 2E_{2^m}(f) \sum_{\nu=2^m}^{2^{m+1}-1} |\gamma_{\nu}(r)| \\ &\quad + \sum_{\nu=2^m}^{2^{m+1}-1} |\gamma_{\nu}(r)| \left| \frac{1}{\pi} \int_{\pi/2^m}^{\pi} \frac{\varphi_m(x; t)}{2 \operatorname{tg} t/2} \sin \nu t dt \right| \leq 12E_{2^m}(f)\delta(m; r). \end{aligned}$$

From this estimate it follows that

$$\sum_{m=0}^{\infty} \theta_m(f; x; r) \leq 12 \sum_{m=0}^{\infty} E_{2^m}(f)\delta(m; r).$$

In an analogous way the following assertion is also established.

Theorem 4. If $\{\gamma_{\nu}(n)\}$ is an arbitrary triangular matrix of numbers, i.e. $\gamma_{\nu}(n) = 0$ ($\nu > n$), then

$$\sum_{\nu=0}^n |\gamma_{\nu}(n)| |f(x) - S_{\nu}(f; x)| \leq 12 \left\{ \sum_{\nu=0}^{m-1} E_{2^{\nu+1}}(f)\delta(\nu; n) + E_{2^m}(f) \left(2^m \sum_{\nu=2^m}^n \gamma_{\nu}^2(n) \right)^{1/2} \right\},$$

where

$$\delta(\nu; n) = \left\{ 2^{\nu} \sum_{k=2^{\nu}}^{2^{\nu+1}-1} \gamma_k^2(n) \right\}^{1/2}, \quad 2^m < n \leq 2^{m+1}.$$

From Theorems 3 and 4 there follows a series of corollaries for classical summability methods. We give one of them.

Corollary. If $\gamma_{\nu}(n) = 1/(n+1)$, $\nu = 0, 1, \dots, n$, $\gamma_{\nu}(n) = 0$ for $\nu > n$, then

$$\frac{1}{n+1} \sum_{\nu=0}^n |f(x) - S_{\nu}(f; x)| \leq \frac{C}{n+1} \sum_{\nu=0}^n E_{\nu}(f). \quad (15)$$

We note that in the case when $E_n(f) = (n+1)^{-\alpha}$, ($0 < \alpha \leq 1$), Aleksich and Kralik ⁽⁴⁾ obtained the estimate

$$\frac{1}{n+1} \sum_{\nu=0}^n |f(x) - S_\nu(f; x)| \leq C \begin{cases} (n+1)^{-\alpha}, & 0 < \alpha < 1, \\ (n+1)^{-1} \ln(n+1), & \alpha = 1, \end{cases}$$

which follows from inequality (15).

The results of the present note, with proofs, were presented by the author at the scientific school on summability theory (Sverdlovsk, 12 VII 1967).

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Note: Figure translations are in progress. See original paper for figures.

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