

CHARACTER SUMS AND PRIMITIVE ROOTS IN FINITE FIELDS

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Abstract

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MATHEMATICS

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CHARACTER SUMS AND PRIMITIVE ROOTS IN FINITE FIELDS

(Presented by Academician I. M. Vinogradov on 2 X 1967)

In the present article, using one example, a new approach is set forth to the problem of estimating character sums and the distribution of primitive roots in finite fields (see (1-6)). The corresponding estimate of the work (5) already holds for $H > p^{1/4+\varepsilon}$, $\varepsilon > 0$ an arbitrarily small number, and any fixed $n \geq 1$. The method of proof of the main theorem is also applicable to more general situations; the theorems of the article can be sharpened and generalized.

Let $n \geq 1$ be an integer, and let

$$F(x) = x^n + a_1 x^{n-1} + \dots + a_n$$

be a polynomial with integer coefficients, irreducible mod p , where p is a prime number, $p \geq p_0 = p_0(F)$. Let, further, θ be a root of the equation

$$F(\theta) = 0$$

and let $G(p^n)$ be the Galois field with basis $\omega_1, \dots, \omega_n$, where $\omega_0 = 1, \omega_1 = \theta, \dots, \omega_n = \theta^{n-1}$. By χ we denote a nonprincipal Dirichlet character of the multiplicative group of the field $G(p^n)$.

If $x \in G(p^n)$, then $x = \omega_1 x_1 + \dots + \omega_n x_n$, where $0 \leq x_i \leq p-1$, $i = 1, 2, \dots, n$. The set of $x \in G(p^n)$ for which $0 \leq \nu_i < x_i \leq \nu_i + H < p$, $i = 1, 2, \dots, n$, will be denoted by $D_1(H) = D_1(H, \nu_1, \dots, \nu_n)$. For $\nu_1 = \dots = \nu_n = 0$, instead of $D_1(H)$ we shall write $D_0(H)$.

The constants in the sign $\ll$ of I. M. Vinogradov and in the symbol O depend, generally speaking, on F .

Theorem 1. For every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$, $p_1 = p_1(\varepsilon)$, such that, for

$$H > p^{1/4+\varepsilon}, \quad p > p_1,$$

the estimate

$$\left| \sum_{x \in D_1(H)} \chi(x) \right| < (Hp^{-\delta})^n$$

holds.

Theorem 2. Under the conditions of Theorem 1, the number of primitive roots lying in $D_1(H)$ is equal to

$$\frac{\varphi(p^n - 1)}{p^n - 1} H^n (1 + O(p^{-n\delta})).$$

The proof of Theorem 2 follows from Theorem 1 in the usual way. For the proof of Theorem 1 we shall need the following two lemmas.

Lemma 1. Let r be an arbitrary positive integer and $0 < h < p$. The following estimate holds:

$$\sum_{\lambda \in D_0(p)} \left| \sum_{z \in D_0(h)} \chi(\lambda + z) \right|^{2r} < (4r)^{r+1} p^n h^{nr} + 2rp^{1/2} h^{2nr}.$$

Lemma 2. Let I be the number of solutions in $G(p^n)$ of the equation

$$xy = x'y',$$

where $x, x' \in D_1(H)$, $y, y' \in D_0(H_1)$, $1 \leq HH_1 \ll p$.

Then for any $\varepsilon_1 > 0$ the estimate holds

$$I \ll (HH_1)^{n(1+\varepsilon_1)}.$$

Lemma 1 and its earlier analogues were proved by H. Davenport, P. Erdős⁽⁷⁾, and D. Burgess⁽³⁾. The proof makes essential use of A. Weil's theorem⁽⁸⁾ on zeros of special zeta-functions. Lemma 2 is a simple generalization of Yu. V. Linnik's lemma (see⁽⁹⁾, Lemma 6).

Proof of Theorem 1. We shall assume that

$$p^{1/4+\varepsilon} < H < p^{1/2+\varepsilon/8},$$

$0 < \varepsilon < 0.01$, since otherwise the theorem follows from⁽⁵⁾. Define the quantities r, δ, H_1, H_2 by the equalities

$$r = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1, \quad \delta = \frac{\varepsilon}{4} \left(r + \frac{n}{2} \right)^{-1}, \quad H_1 = Hp^{-1/2r-n\delta}, \quad H_2 = p^{1/2r},$$

and let $y \in D_0(H_1)$, $z \in D_0(H_2)$. Then the following relation is valid (see^(10,11)):

$$\sum_{x \in D_1(H)} \chi(x) = \sum_{x \in D_1(H)} \chi(x + yz) + O((Hp^{-\delta})^n). \quad (1)$$

Indeed, if

$$y = \sum_{i=1}^n \omega_i y_i, \quad z = \sum_{j=1}^n \omega_j z_j, \quad \omega_i \omega_j = \sum_{k=1}^n \omega_k d_{ijk},$$

then

$$yz = \sum_{k=1}^n \omega_k t_k, \quad (2)$$

where

$$t_k = \sum_{i,j=1}^n y_i z_j d_{ijk} \ll H_1 H_2 = Hp^{-n\delta}.$$

Therefore we have

$$\begin{aligned} \sum_{x \in D_1(H)} \chi(x) &= \sum_{\nu_1 < x_1 \leq \nu_1 + H} \cdots \sum_{\nu_n < x_n \leq \nu_n + H} \chi(\omega_1 x_1 + \cdots + \omega_n x_n) = \\ &= \sum_{\nu_1 < x_1 \leq \nu_1 + H} \cdots \sum_{\nu_{n-1} < x_{n-1} \leq \nu_{n-1} + H} \left(\sum_{\nu_n < x_n \leq \nu_n + t_n + H} \chi(\omega_1 x_1 + \cdots + \omega_n x_n) \right. \\ &\quad \left. + \sum_{\nu_n < x_n \leq \nu_n + t_n} \chi(\omega_1 x_1 + \cdots + \omega_n x_n) - \sum_{\nu_n + H < x_n \leq \nu_n + t_n + H} \chi(\omega_1 x_1 + \cdots + \omega_n x_n) \right) = \\ &= \sum_{\nu_1 < x_1 \leq \nu_1 + H} \cdots \sum_{\nu_n < x_n \leq \nu_n + H} \chi(\omega_1 x_1 + \cdots + \omega_n (x_n + t_n)) + 2\theta_n H^{n-1} t_n = \cdots = \\ &= \sum_{\nu_1 < x_1 \leq \nu_1 + H} \cdots \sum_{\nu_n < x_n \leq \nu_n + H} \chi(\omega_1 (x_1 + t_1) + \cdots + \omega_n (x_n + t_n)) + 2\theta_1 H^{n-1} t_1 + \cdots + 2\theta_n H^{n-1} t_n, \end{aligned}$$

where $|\theta_1| \leq 1, \dots, |\theta_n| \leq 1$. Hence, from (2), (1) follows.

Summing (1) over all $y \in D_0(H_1)$, $z \in D_0(H_2)$, we obtain

$$\sum_{x \in D_1(H)} \chi(x) = W + O((Hp^{-\delta})^n), \quad (3)$$

where

$$W = (H_1 H_2)^{-n} \sum_{x \in D_1(H)} \sum_{y \in D_0(H_1)} \sum_{z \in D_0(H_2)} \chi(x + yz).$$

Let us consider the sum W . Using the multiplicativity of χ , we obtain

$$|W| \ll (H_1 H_2)^{-n} \sum_{\lambda} I(\lambda) \left| \sum_{z \in D_2(H_2)} \chi(\lambda + z) \right|, \quad (4)$$

where $I(\lambda)$ is the number of solutions of the equation $xy^{-1} = \lambda$, $x \in D_1(H)$, $y \in D_0(H_1)$. Since

$$\sum_{\lambda} I(\lambda) = (H H_1)^n, \quad \sum_{\lambda} I^2(\lambda) = I,$$

raising (4) to the power $2r$ and successively applying Hölder's and Cauchy's inequalities gives

$$|W|^{2r} \ll (H_1 H_2)^{-2nr} \left(\sum_{\lambda} I(\lambda) \right)^{2(r-1)} \sum_{\lambda} I^2(\lambda) \sum_{\lambda} \left| \sum_{x \in D_0(H_2)} \chi(\lambda + z) \right|^{2r} \ll$$

$$\ll (H_1 H_2)^{-2nr} (H H_1)^{2n(r-1)} (H H_1)^{n(1+\varepsilon_1)} \sum_{\lambda \in D_0(p)} \left| \sum_{z \in D_0(H_2)} \chi(\lambda + z) \right|^{2r}.$$

Applying the estimate of Lemma 1 to the last double sum, after simple computations we arrive at the estimate

$$|W| \ll (H p^{-\delta})^n.$$

The assertion of the theorem follows from this inequality and (3).

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