

ELEMENTS OF INFINITE FILTRATION IN (K) -THEORY

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Abstract

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MATHEMATICS

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ELEMENTS OF INFINITE FILTRATION IN K -THEORY

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By a CW -complex in this paper we mean locally finite CW -complexes in the sense of J. H. C. Whitehead ⁽¹⁾. In ⁽²⁻⁴⁾, the functors $\mathcal{K}^*(X)$ and $k^*(X)$ were defined on the category of CW -complexes. M. F. Atiyah and F. Hirzebruch ⁽³⁾ proved that, for any compact connected Lie group G , the completed ring of unitary representations $\hat{R}(G)$ is isomorphic to the ring $\mathcal{K}^0(BG)$, and conjectured that $\mathcal{K}^*(BG) = k^*(BG)$, i.e., that the ring $\hat{R}(G)$ is isomorphic to the set of homotopy classes of maps of the space BG into the space BU . In the present paper we prove this conjecture (Theorem 3).

I. In ^(9,10) it was shown that k is a generalized cohomology theory, and that the topology in the ring $k(X)$ generated by the filtration by finite-dimensional skeleta may be non-Hausdorff. In other words, there may exist maps of the complex X into BU that are homotopic to the constant map on each skeleton and are not homotopic to the constant map on the whole complex X . It is also proved there that $\mathcal{K}^*(X) = k^*(X)/\{\bar{0}\}$.

Theorem 1. Let X be a locally finite CW -complex. In order that $\mathcal{K}^0(X) = k^0(X)$, it is necessary and sufficient that for every element $\alpha \in H^{\text{odd}}(X, Q)$ there exist a map $f : SX \rightarrow BU$ and a number N such that $f^*(\text{ch } \eta) = N(s\alpha) +$ terms of higher dimension, where η is the canonical element of $k^0(BU)$. An analogous condition holds for the equality $\mathcal{K}^1(X) = k^1(X)$.

In applications the following formulation is useful:

Theorem 1'. In order that $\mathcal{K}^0(X) = k^0(X)$, it is necessary and sufficient that, in the spectral sequence (see ^(3,4)) converging to $\mathcal{K}^*(X)$ and whose term E_2 is equal to $H^*(X, Z)$, for every element $\alpha \in E_2$ of infinite order there exist a number N such that $N\alpha$ is a cycle of all differentials.

Corollary 1. If $H^{\text{odd}}(X, Q) = 0$, then $\mathcal{K}^0(X) = k^0(X)$; correspondingly, if $H^{\text{ev}}(X, Q) = 0$, then $\mathcal{K}^1(X) = k^1(X)$.

Corollary 2. Let X be a $2l$ -connected CW -complex such that the group $H^{2(n+l)}(X, Z)$ has no elements annihilated by the number $(n-1)!$. Then

$\mathcal{K}^0(X) = k^0(X)$. It is easy to formulate the corresponding condition for the equality $\mathcal{K}^1(X) = k^1(X)$.

Corollary 3. Let $\pi : \hat{X} \rightarrow X$ be a finite regular covering. $\mathcal{K}^i(\hat{X}) = k^i(X)$ only when $\mathcal{K}^i(X) = k^i(X)$, $i \in \mathbb{Z}_2$.

II. Applications.

Theorem 2. Let X be a complex such that $H^*(\Omega X, Q)$ is a finitely generated Q -module. Then $\mathcal{K}^0(X) \approx k^0(X)$.

Proof. According to a well-known theorem of H. Hopf ⁽⁵⁾, under the hypotheses of the theorem the ring $H^*(\Omega X, Q)$ is an exterior algebra on odd-dimensional primitive generators, if X is simply connected. Consequently, in this case the ring $H^*(X, Q)$ is a polynomial algebra on even-dimensional generators. Thus, for the complex X the conditions are satisfied

Corollary 1. Therefore $\mathcal{K}^0(X) = \varprojlim (k^0(X^n)) = k^0(X)$. In the general case one must apply Corollary 3, since, by the hypothesis of the theorem, $\pi_1(X)$ is a finite group.

Theorem 3. Let G be a compact Lie group. Then the rings $\mathcal{K}^*(BG)$ and $k^*(BG)$ are isomorphic.

Proof. In view of Corollary 3 it suffices to consider only a connected group G . We note that the group G is homotopy equivalent to the space ΩBG . Hence, applying Theorem 2, we obtain the equality $\mathcal{K}^0(BG) = k^0(BG)$.

We shall now prove that for any element $a \in H^{2i}(BG, Q)$ there exist an element $\eta \in \mathcal{K}^0(BG)$ and a number N such that $\text{ch } \eta = Na +$ terms of higher dimension. Let $\rho : T \subset G$ be a maximal torus. Then we have the commutative diagram

$$\begin{array}{ccc} \mathcal{K}^0(BT) & \xleftarrow{\rho^*} & \mathcal{K}^0(BG) \\ \text{ch} \downarrow & & \downarrow \text{ch} \\ H^{**}(BT, Q) & \xleftarrow{\rho^{**}} & H^{**}(BG, Q) \end{array}$$

The mapping ρ^{**} is a monomorphism (6). By Corollary 2, for the space $BT = \prod_{i=1}^r CP^\infty$ the hypotheses of Theorem 1 are satisfied. Therefore, for any element

$a \in H^{2i}(BG, Q)$, there exist such an N and such an element $y \in \mathcal{K}^0(BT)$ that $\text{ch } y = N\rho^{**}a +$ terms of higher dimension. If Γ is the Weyl group of the group G , then the element $z = \sum_{\gamma \in \Gamma} \gamma(y)$ is invariant with respect to the Weyl group,

and then, by Theorem II.4.4 of (3), there exists an element $x \in \mathcal{K}^0(BG)$ such that $\rho^*(x) = z$. Then $\text{ch } z = N \cdot \text{ord } \Gamma \cdot a +$ terms of higher dimension. Thus we have proved that the hypotheses of Theorem 1 are satisfied for the space BG .

If the topological group G is not compact, then, generally speaking, $\mathcal{K}^*(BG)$ is not isomorphic to $k^*(BG)$, for example in the case $G = CP^\infty$ (see (9, 10)). However, Theorem 1 implies the following.

Theorem 4. *Let $BO(n, \dots, \infty)$ ($BU(n, \dots, \infty)$) be the n -connected space of the classifying space of the infinite-dimensional orthogonal group O (unitary group U). Then*

$$\mathcal{K}^*(BO(n, \dots, \infty)) = k^*(BO(n, \dots, \infty)), \quad \mathcal{K}^*(BU(n, \dots, \infty)) = k^*(BU(n, \dots, \infty)).$$

The computation of these rings may be found in (10).

III. Proof of Theorem 1. We shall outline the plan of the proof. Let

$$(V)_n \rightarrow (V)_{n-1}$$

be a sequence of killing spaces of a CW-complex V and of fibrations with fiber $K(\pi_{n-1}(V), n-2)$.

Lemma 1. *Suppose there is given such a sequence of mappings $\{\varphi_n : X \rightarrow (V)_n\}$ that*

$$\varphi_n \simeq f_{n+1} \cdot \varphi_{n+1}.$$

Then all the mappings φ_n are homotopic to the constant mapping.

Lemma 2. *Let X and V be CW-complexes such that, for any $i > 0$,*

$$H^i(X, Q) \otimes \pi_{i+1}(V) = 0.$$

If the complex V is an H -space, then the homomorphism

$$\pi : [X, V]_0 \rightarrow \varprojlim_n [X^n, V]$$

is an isomorphism.

Here $[X, V]_0$ is the group of homotopy classes of mappings with fixed base point, and X^n are the n -dimensional skeleta of the complex X .

Proof. It is enough to prove the equality $\ker \pi = 0$. Let $\varphi : X \rightarrow (V)_n$ be a mapping homotopic to the constant mapping on each skeleton X^k . We shall prove that there exists a mapping $\psi : X \rightarrow (V)_{n+1}$ such that $f_{n+1} \cdot \psi \simeq \varphi$, and ψ is also homotopic to the constant mapping on each skeleton X^k . Consider the fibration

$$(V)_{n+1} \xrightarrow{K(\pi_n(V), n-1)} (V)_n.$$

It is clear that there exists a lifting $\chi : X \rightarrow (V)_{n+1}$, i.e. $f_{n+1} \cdot \chi = \varphi$. As follows ...

it is known that in this case every other lifting is uniquely determined by the homotopy class of a map of the space X into the fiber $K(\pi_n(V), n-1)$. In our situation there is a finite number of such homotopy classes. Consequently,

among these liftings there is at least one lifting ψ , homotopic to the map into a point on every skeleton X^k . By induction we construct a sequence of maps $\varphi_n : X \rightarrow (V)_n$ satisfying the conditions of Lemma 1.

Lemma 3. *If $H^*(X, \mathbf{Z})$ has no torsion, then $\mathcal{K}^*(X) = k^*(X)$.*

It is easy to verify that in the given situation the conditions of Lemma 1 can be made to hold.

It follows from Lemma 3 that for the spaces BU , $BU/[BU]^n$, where $[BU]^n$ is the $2n$ -dimensional skeleton of BU ,

$$Y = \lim_{\substack{\longrightarrow \\ N}} \prod_{i=1}^N BU/[BU]^{n_i}, \quad n_i \rightarrow \infty,$$

the groups

$$k^1 = \mathcal{K}^1 = 0.$$

If X satisfies the conditions of Theorem 1, then there exists a map $f : X \rightarrow Y$ such that $f : H^{2i}(Y, Q) \rightarrow H^{2i}(X, Q)$ is an epimorphism for all i . Then $H^{2i+1}(Y, X; Q) = 0$. By Lemma 2, $\mathcal{K}^0(Y/X) = k^0(Y/X)$. Consider the exact sequence of the pair (Y, X) :

$$k^0(Y/X) \leftarrow k^1(X) \leftarrow k^1(Y) = 0.$$

Since the topology in the ring $k^0(Y/X)$ is Hausdorff, we have $k^1(X) = \mathcal{K}^1(X)$ (recall that $k^1(X) = \mathcal{K}^1(X)/\{0\}$). Replacing X by SX , we obtain the isomorphism $\mathcal{K}^0(X) = k^0(X)$.

Now suppose there is an element $a \in H^{2i}(X, Q)$ such that one cannot find $y \in \mathcal{K}^0(X)$ with $\text{ch } y = Na +$ terms of higher dimension. Consider the pair (X, X^{2i}) , and for it the commutative diagram:

$$\begin{array}{ccccccc} \leftarrow & k^1(X) & \xleftarrow{\pi^*} & k^1(X/X^{2i}) & \xleftarrow{\delta^*} & k^0(X^{2i}) & \xleftarrow{i^*} k^0(X) \leftarrow \\ & \downarrow \text{ch} & & & & \downarrow \text{ch} & \downarrow \text{ch} \\ \leftarrow & H^{odd}(X/X^{2i}, Q) & \xleftarrow{\delta^*} & H^{ev}(X^{2i}, Q) & \xleftarrow{i^*} & H^{ev}(X, Q) & \leftarrow \end{array}$$

It is clear that $i^*(a) \neq 0$. Since X^{2i} is a finite complex, there exists an element $y_1 \in \mathcal{K}^0(X^{2i})$ such that $\text{ch } y_1 = Ni^*a$. By assumption, $y_1 \notin \text{Im } i^*$. Then $z = \delta^*(y_1) \neq 0$ and $\text{ch } z = 0$. Consider the subgroup $S \subset k^1(X/X^{2i})$ generated by the element z . It can be shown ⁽¹⁰⁾ that if $\mathcal{K}^1(X) = k^1(X)$, then $\mathcal{K}^1(X/X^{2i}) \simeq k^1(X/X^{2i})$, i.e. $S \cap \{0\} = 0$. Since $\text{ch } z = 0$, the completion $\widehat{S}$ of the group S in the topology of the ring $k^1(X/X^{2i})$ is continual. Hence the set $\text{Im } \delta^* \subset$

$k^1(X/X^{2i})$ is not closed (since $k^0(X^{2i})$ is a finitely generated group). Since $\text{Im } \delta^* = \ker \pi^*$, the set $\{0\} \subset k^1(X)$ is not closed, which was required to be proved.

IV. J. Adams and G. Walker ⁽⁸⁾ constructed an example of a map f of a complex X into a complex Y (Y not locally finite, but finite-dimensional), such that the restriction of f to any skeleton of the space X is homotopic to a map into a point, while f itself is not homotopic to a map into a point. We shall show that such examples also arise in k -theory.

As is known ⁽⁷⁾, for any prime p the group $\pi_{2p}^p(S^3) = \mathbf{Z}_p$, where π_k^p is the p -component of the group π_k . Therefore there exists a complex

$$X_p = S^3 \cup_{\varphi} (D^{2p+1} \cup_p D^{2p+2}),$$

where $\varphi : S^{2p} \cup_p D^{2p+1} \rightarrow S^3$ is the map corresponding to a generator of the group $\pi_{2p}^p(S^3)$, $\tilde{\varphi} : S^{2p} \rightarrow S^3$. The cohomologies $H^*(X_p, \mathbf{Z})$ have the form

$$H^3(X_p, \mathbf{Z}) = \mathbf{Z}$$

(a -generator),

$$H^{2p+2}(X_p, \mathbf{Z}) = \mathbf{Z}_p$$

(b -generator). The integral operation (βP^1) carries the element a into b . Let X be the complex obtained by identifying in all the complexes X_p the spheres S^3 with one another. The resulting complex is locally finite. In an analogous way one may glue a complex using any sequence of primes $\{p_i\}$. Fix some infinite sequence of primes not exhausting

all primes, and denote the corresponding complex by Y . It is easy to see that the complex Y is embedded in X . We compute $k^*(X, \mathbf{Z}_p)$, $\mathcal{K}^*(X)$, $\mathcal{K}^*(Y)$, $\mathcal{K}^*(X/Y)$. For the definition of the functor $k^*(X, \mathbf{Z}_p)$, see (9).

Consider the spectral sequence induced by the filtration by skeleta and, as is easy to show, strongly convergent to $\mathcal{K}^*(X)$. The term

$$E_2^{l,q} = H^l(X, k^q(*)), \quad q \in \mathbf{Z}_2.$$

All differentials d_r have finite order, and the first nontrivial p -component occurs for the differential d_{2p-1} and is equal to the integral cohomology operation (βP^1) (see (4)). From this it is easy to obtain that

$$E_{\infty}^*(X) = E_{\infty}^*(Y) = 0.$$

Thus $\mathcal{K}^*(X) = \mathcal{K}^*(Y) = 0$. The complex X/Y , however, is a bouquet of some p -spheres, and therefore

$$\mathcal{K}^*(X/Y) \neq 0.$$

To compute $k^*(X, \mathbf{Z}_p)$, we apply the analogous spectral sequence, strongly convergent to

$$\mathcal{K}^*(X, \mathbf{Z}_p) = k^*(X, \mathbf{Z}_p). \quad (10)$$

The term

$$E_2^{l,q} = H^l(X, k^q(*, \mathbf{Z}_p)), \quad q \in \mathbf{Z}_2.$$

Let us write down the nonzero terms:

$$E_2^{3,0} = \mathbf{Z}_p, \quad E_2^{2p+1,0} = \mathbf{Z}_p, \quad E_2^{2p+2,0} = \mathbf{Z}_p.$$

The differential

$$d_{2p-1} : E_{2p-1}^{3,0} \rightarrow E_{2p-1}^{2p+2,0}$$

is nontrivial; consequently,

$$E_\infty^{2p+1,0} = E_{2p}^{2p+1,0} = \mathbf{Z}_p$$

is the only nontrivial group. Thus

$$k^*(X, \mathbf{Z}_p) = \mathbf{Z}_p.$$

From the computations given above we draw several consequences:

1. For infinite complexes the groups $\mathcal{K}^*(X)$ and $\mathcal{K}^*(X, \mathbf{Z}_p)$ are not related by the Künneth formula.
2. For the $\mathcal{K}^*$ -functor there is no exact sequence of a pair; moreover, the closure of the image is, in general, not equal to the kernel. Indeed, the pair (X, Y) constructed above has the sequence

$$\leftarrow \mathcal{K}^*(Y) \leftarrow \mathcal{K}^*(X) \leftarrow \mathcal{K}^*(X/Y) \leftarrow \mathcal{K}^*(Y) \leftarrow$$

$$\begin{array}{cccc} \parallel & \parallel & \nearrow & \parallel \\ 0 & 0 & 0 & 0 \end{array}$$

which, of course, is not exact.

3. Since $k^*(X, \mathbf{Z}_p) \neq 0$, it follows from the Künneth formula (see (10)) that $k^*(X) \neq 0$; on the other hand $\mathcal{K}^*(X) = 0$, hence

$$k^*(X) = \{\bar{0}\},$$

i.e., all elements of $k^*(X) = [S^{kX}, BU]$ are represented by maps

$$S^{kX} \rightarrow BU \quad (k \geq 0),$$

homotopic to the constant map on each skeleton, but not homotopic to the constant map on the whole complex.

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