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Abstract

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HYDROMECHANICS

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ON ONE PERIODIC SOLUTION OF THE DIFFUSION EQUATION WITH A NONLINEAR BOUNDARY CONDITION

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In works (1-5) the problem was considered of finding a periodic solution of the one-dimensional diffusion equation $\partial u / \partial t = D \partial^2 u / \partial x^2$ in the half-infinite region $x \geq 0$ with the nonlinear boundary condition $\partial u / \partial x|_{x=0} = F[u(0, t)]$ and with an a priori unknown period T , depending on the solution. (Here $F(u)$ is a multi-valued S-shaped function, and when the value $u(0, t)$, increasing, reaches a fixed value u_+ , a jump occurs from the branch $F_1(u) < 0$ to the branch $F_2(u) > 0$; when $u(0, t)$, decreasing, reaches u_- , a jump occurs from $F_2(u)$ to $F_1(u)$.) It is shown below that, under certain assumptions, there exists at least one solution of this problem.

For the case $F_2(u) = -F_1(-u)$, $u_- = -u_+$, in paper (4) the solution of this problem was reduced to the integration of the nonlinear integral equation

$$\lambda U = AU, \quad (1)$$

where

$$AU = -\frac{1}{\sqrt{\pi}} \left\{ \int_0^\tau \frac{F[U(\sigma)] d\sigma}{\sqrt{\tau - \sigma}} + \int_0^{1/2} F[U(\sigma)] Q(\tau - \sigma) d\sigma \right\}, \quad 0 \leq \tau \leq \frac{1}{2}, \quad (2)$$

$$U(\tau) = u(0, t)/u_+, \quad \tau = t/T, \quad \lambda = u_+/\mu\sqrt{DT},$$

$$F(U) = \mu F_1(u) < 0, \quad Q(z) = \sum_{n=1}^{\infty} \left\{ -\frac{1}{\sqrt{n - \frac{1}{2} + z}} + \frac{1}{\sqrt{n + z}} \right\}.$$

The function $Q(z)$ is defined for $-\frac{1}{2} \leq z \leq \frac{1}{2}$, and $Q(z) < 0$,

$$Q'(z) > 0, \quad \lim_{z \rightarrow -1/2} Q(z) = \infty, \quad \int_{-1/2}^{1/2} Q(z) dz$$

converges.

We shall assume that, for values of U close to 1, the function $F(U)$, defined by formula (2), admits the asymptotic representation

$$F(U) = E + N(1 - U)^r + \dots \quad (1/2 < r < 1, E < 0, N < 0). \quad (3)$$

By virtue of the conditions imposed on the S-shaped function, the operator AU must be such that the function

$$y(\tau) = AU, \quad (4)$$

where the operator U is given by formula (2), attains its maximum value m only for $\tau = 0$ and its minimum value $-m$ only for $\tau = \frac{1}{2}$. It is easy to verify that from formulas (2) and (4) it follows that $y(0) = -y(1/2)$.

This requirement ($-m \leq y(\tau) \leq m$ for $0 \leq \tau \leq \frac{1}{2}$), as can be seen, will in any case be fulfilled if the function $F(U)$ has the form

$$F(U) = -F_0 + \varkappa \varphi(U) \quad (F_0 > 0, \|\varphi(U)\| = 1) \quad (5)$$

for sufficiently small $\varkappa$.

We shall show that the operator AU is completely continuous in the space of continuous functions C . By Arzelà's theorem, for this it is sufficient to show that: 1) $\lim AU_n = AU$ ($U_n \rightarrow U$); 2) $|y(\tau)| \leq M$; 3) for any $\varepsilon > 0$ there exists a $\delta(\varepsilon)$ such that $|y(\tau) - y(\tau')| \leq \varepsilon$ if $|\tau - \tau'| < \delta(\varepsilon)$, where y is given by formula (4). Here $U(\tau)$ is any continuous function such that $\|U(\tau)\| = 1$, and the value -1 is attained by this function only for $\tau = 0$, and the value 1 only for $\tau = 1/2$.

Condition 1) is fulfilled by virtue of the continuity of the function $F(U)$

$$\lim F(U_n) = F(U) \quad (U_n \rightarrow U). \quad (6)$$

Introduce the notation

$$Q_1(z) = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n+z}} - \frac{1}{\sqrt{n+1/2+z}} \right) \quad \left(Q(z) = -\frac{1}{\sqrt{1/2+z}} + Q_1(z) \right). \quad (7)$$

For $-1/2 \leq z \leq 1/2$, $Q_1(z) > 0$, $Q'_1(z) < 0$, and the operator AU , by virtue of (2) and (7), takes the form

$$AU = -\frac{1}{\sqrt{\pi}} \left\{ \int_0^\tau \frac{F[U(\sigma)] d\sigma}{\sqrt{\tau-\sigma}} - \int_0^{1/2} \frac{F[U(\sigma)] d\sigma}{\sqrt{1/2+\tau-\sigma}} + \int_0^{1/2} F[U(\sigma)] Q_1(\tau-\sigma) d\sigma \right\}. \quad (8)$$

From (8) we have

$$\|AU - AU_n\| \leq \frac{1}{\sqrt{\pi}} \max |F[U(\sigma)] - F[U_n(\sigma)]| \{2\sqrt{2} + \frac{1}{2}Q_1(-1/2)\}$$

and consequently, by (6), condition 1) is fulfilled.

We shall show that condition 2) is also fulfilled. For every function $y(\tau)$ we have $\|y(\tau)\| = -y(0) = y(1/2)$, and consequently, by virtue of (2),

$$\|y(\tau)\| = \frac{1}{\sqrt{\pi}} \int_0^{1/2} F[U(\sigma)] Q(-\sigma) d\sigma.$$

Using the inequalities $F_{\min} \leq F[U(\sigma)] \leq F_{\max}$, we obtain for all $y(\tau)$

$$m' \leq \|y(\tau)\| \leq M' \quad \left(M' = \frac{F_{\min}}{\sqrt{\pi}} \int_0^{1/2} Q(-\sigma) d\sigma, \quad m' = \frac{F_{\max}}{\sqrt{\pi}} \int_0^{1/2} Q(-\sigma) d\sigma \right). \quad (9)$$

Now consider condition 3). By virtue of (2) we have (for $\tau > \tau'$)

$$\begin{aligned} y(\tau) - y(\tau') &= AU(\tau) - AU(\tau') = \\ &= -\frac{1}{\sqrt{\pi}} \left[\int_0^{\tau'} F[U(\sigma)] \left[\frac{1}{\sqrt{\tau-\sigma}} - \frac{1}{\sqrt{\tau'-\sigma}} \right] d\sigma + \int_{\tau'}^\tau \frac{F[U(\sigma)] d\sigma}{\sqrt{\tau-\sigma}} \right. \\ &\quad + \int_0^{1/2} F[U(\sigma)] \left\{ -\frac{1}{\sqrt{1/2+\tau-\sigma}} + \frac{1}{\sqrt{1/2+\tau'-\sigma}} \right\} d\sigma \\ &\quad \left. + \int_0^{1/2} F[U(\sigma)] \{Q_1(\tau-\sigma) - Q_1(\tau'-\sigma)\} d\sigma \right]. \quad (10) \end{aligned}$$

Further,

$$y(\tau) - y(\tau') \leq \frac{1}{\sqrt{\pi}} [-F_{\min}] \left[\int_0^{\tau'} \left(\frac{1}{\sqrt{\tau' - \sigma}} - \frac{1}{\sqrt{\tau - \sigma}} \right) d\sigma + 2\sqrt{\tau - \sigma} \right]_{\tau'}^{\tau} \\ + \int_0^{1/2} \left(\frac{1}{\sqrt{1/2 + \tau' - \sigma}} - \frac{1}{\sqrt{1/2 + \tau - \sigma}} \right) d\sigma + \{Q_1(-1/2) - Q_1(0)\}(\tau - \tau') \Bigg]. \quad (11)$$

Thus,

$$|y(\tau) - y(\tau')| \leq M\sqrt{\tau - \tau'}, \quad \varepsilon = M\sqrt{\delta}$$

$$\left(M = K + \frac{L}{\sqrt{2}}, \quad K = \frac{-4F_{\min}}{\sqrt{\pi}}, \quad L = -\frac{F_{\min}}{\sqrt{\pi}} [Q_1(-1/2) - Q_1(0)] \right). \quad (12)$$

Let us now consider equation (1). As σ varies from 0 to 1/2, the function U in expression (2) varies from -1 to 1 ; at the same time AU varies from $-m$ to m , and for the solution $U(\tau)$ of equation (1) we have $\lambda U(0) = -m$, $\lambda U(1/2) = m$, or, since $U(0) = -1$, $U(1/2) = 1$, $\lambda = m = \|AU\| = y(1/2)$. Thus, instead of equation (1) one may consider the equation

$$U = BU, \quad (13)$$

where the notation

$$BU = AU/\|AU\| \quad (\|BU\| = 1), \quad (14)$$

has been introduced, i.e. the operator BU maps the sphere $\|U\| = 1$ into itself.

The operator BU is completely continuous; consequently, there exists a finite ε_0 -net $\eta_1, \eta_2, \dots, \eta_p$, i.e. for any function U there is at least one element η_j such that $\|BU - \eta_j\| < \varepsilon_0$ (where, for example, $\varepsilon_0 = 5\varepsilon/m'$, ε is given by formula (12), m' —by (9)⁶).

Now we construct a Schauder projection operator P_{n+1} of the set BU onto E^{n+1} such that

$$\|P_{n+1}BU - BU\| < \varepsilon_0 \quad (\|U\| = 1). \quad (15)$$

We define the operator $P_{n+1}z$ by formula (7)

$$P_{n+1}z = \sum_{j=1}^p \mu_j(z) \eta_j / \sum_{j=1}^p \mu_j(z) \quad (z \in BL),$$

$$\mu_j(z) = \begin{cases} \varepsilon_0 - \|z - \eta_j\|, & \text{if } \|z - \eta_j\| \leq \varepsilon_0, \\ 0, & \text{if } \|z - \eta_j\| \geq \varepsilon_0, \end{cases} \quad (j = 1, 2, \dots, p), \quad (16)$$

where $\eta_j \in E^{n+1}$ are the elements of the finite ε_0 -net. It is easy to see that (16) implies that

$$\|P_{n+1}BU - BU\| < \varepsilon_0. \quad (17)$$

Let us now consider the operator

$$\Phi_{n+1}U = P_{n+1}BU \quad (18)$$

on the intersection of L with E^{n+1} , $L \cap E^{n+1}$, where $\|U\| = 1$, $U(0) = -1$, $U(1/2) = 1$.

Since the operator BU is completely continuous, the set BU is compact for every continuous function U , including the case when $U \in L \cap E^{n+1}$.

By Brouwer's theorem, under a continuous mapping of a bounded closed convex body of the $(n+1)$ -dimensional Banach space E into itself there exists a fixed point ^(7,9)

$$\Phi_{n+1}U = U. \quad (19)$$

From formulas (19), (18), and (17) we obtain the inequality

$$\|U - BU\| < \varepsilon_0. \quad (20)$$

Thus, the operator BU has "almost eigenvectors" on the sphere L , $\|U\| = 1$. From the existence of almost eigenvectors there follows the theorem ⁽⁷⁾: let B be a completely continuous operator defined on the boundary L of a bounded domain containing the zero θ of the space E .

Let $\|BU\| \geq a > 0$ ($U \in L$). Then the operator B has at least one eigenvector in L . In the case considered by us, $\|BU\| = 1$ by virtue of (14); hence the theorem is true.

Let U_0 be the solution found of equation (13), $U_0 = BU_0$. By virtue of (14), here $BU_0 = AU_0/\|AU_0\|$, and it is clear that $\lambda = \|AU_0\|$; from (2) we find the period of the self-oscillation

$$T = u_+^2 / \mu^2 D \lambda^2. \quad (21)$$

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