

ANALOGS OF STIRLING' S FORMULA

V. M. Kalinin

1968

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196801.46631>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.5 + 517.6

MATHEMATICS

V. M. KALININ

ANALOGS OF STIRLING' S FORMULA

(Presented by Academician Yu. V. Linnik on 9 X 1967)

Stirling's formula gives an asymptotic expansion of the gamma function $\Gamma(1+x)$ as $x \rightarrow \infty$ in inverse powers of x :

$$\Gamma(1+x) = \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \exp \left\{ \sum_{j=1}^{\nu-1} \frac{B_{j+1}}{j(j+1)x^j} + O\left(\frac{1}{x^\nu}\right) \right\},$$

$$\Gamma(1+x) = \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left\{ 1 + \sum_{j=1}^{\nu-1} \frac{b_j}{x^j} + O\left(\frac{1}{x^\nu}\right) \right\},$$

$$\frac{1}{\Gamma(1+x)} = \frac{1}{\sqrt{2\pi x}} \left(\frac{e}{x}\right)^x \left\{ 1 + \sum_{j=1}^{\nu-1} (-)^j \frac{b_j}{x^j} + O\left(\frac{1}{x^\nu}\right) \right\},$$

where the coefficients b_j can be found from the recurrence relations

$$b_j = \frac{1}{j} \sum_{k=0}^{j-1} b_k B_{j-k+1}, \quad b_0 = 1.$$

Here B_{j-k+1} are Bernoulli numbers. (Somewhat different recurrence equations are indicated in (1).) The Bernoulli numbers are easily computed recursively:

$$B_j = (-)^j j \sum_{k=0}^{j-1} \frac{B_k C_{j-1}^k}{j-k+1}, \quad B_1 = -\frac{1}{2}.$$

In applications (for example, in probability theory) it is often convenient to expand the gamma function in powers of a quantity different from x . The following theorems introduce into Stirling's formula an arbitrary parameter θ , by the choice of which one can obtain the necessary expansions.

Theorem 1. For $x > -1$, $\nu = 2, 3, \dots$, and arbitrary $\theta > -x$

$$\Gamma(1+x) = \sqrt{2\pi} (x+\theta)^{x+1/2} \exp \left\{ -(x+\theta) + \sum_{j=1}^{\nu-1} \frac{B_{j+1}(\theta)}{j(j+1)(x+\theta)^j} + R_\nu \right\},$$

where

$$R_\nu = -\frac{1}{\nu} \int_0^1 B_\nu(t) \zeta(\nu, x+t) dt + \frac{1}{\nu} \int_1^\theta \frac{B_\nu(t)}{(x+t)^\nu} dt,$$

$$\zeta(\nu, x) = \sum_{k=1}^{\infty} \frac{1}{(x+k)^\nu},$$

$B_{j+1}(\theta)$ are Bernoulli polynomials. They may be found, for example, from the following recurrence formulas:

$$(-)^j \frac{B_j(\theta)}{j!} = \sum_{k=0}^{j-1} (-)^k \frac{B_k(\theta)}{k!} \frac{(\theta-1)^{j-k+1} - \theta^{j-k+1}}{(j-k+1)!}, \quad B_0(\theta) = 1.$$

The proof of Theorem 1 is close to the proof of Lemma 1 of paper (2); the difference is that in Taylor's formula, on which the summation theorem is based, the remainder term is taken in integral form.

Theorem 2. As $x \rightarrow \infty$, $\nu = 1, 2, \dots$, uniformly with respect to arbitrary $\theta = O(1)$, the expansions

$$\Gamma(1+x) = \sqrt{2\pi} (x+\theta)^{x+1/2} \exp \left\{ -(x+\theta) + \sum_{j=1}^{\nu-1} \frac{B_{j+1}(\theta)}{j(j+1)(x+\theta)^j} + O\left(\frac{1}{(x+\theta)^\nu}\right) \right\},$$

$$\Gamma(1+x) = \sqrt{2\pi} (x+\theta)^{x+1/2} e^{-(x+\theta)} \left\{ 1 + \sum_{j=1}^{\nu-1} \frac{b_j(\theta)}{(x+\theta)^j} + O\left(\frac{1}{(x+\theta)^\nu}\right) \right\},$$

$$\frac{1}{\Gamma(1+x)} = \frac{e^{x+\theta}}{\sqrt{2\pi} (x+\theta)^{x+1/2}} \left\{ 1 + \sum_{j=1}^{\nu-1} (-1)^j \frac{b_j(1-\theta)}{(x+\theta)^j} + O\left(\frac{1}{(x+\theta)^\nu}\right) \right\},$$

hold, where $b_j(\theta)$ are polynomials of degree $2j$, satisfying the recurrence relations

$$b_j(\theta) = \frac{1}{j} \sum_{k=0}^{j-1} \frac{b_k(\theta) B_{j-k+1}(\theta)}{j-k+1}, \quad b_0(\theta) = 1,$$

$$\Delta b_j(\theta) = \theta b_{j-1}(1 + \theta).$$

In particular, from Theorem 2 we find, as $x \rightarrow \infty$,

$$\Gamma(1+x) = [x]! x^{\{x\}} \exp \left\{ \sum_{j=1}^{\nu-1} \frac{B_{j+1} - B_{j+1}(\{x\})}{j(j+1)x^j} + O\left(\frac{1}{x^\nu}\right) \right\},$$

(here $[x]$ and $\{x\}$ are the integer and fractional parts of x).

As $n \rightarrow \infty$ and $x = O(1)$, the following equality holds uniformly with respect to x :

$$\Gamma(x) = \frac{\sqrt{2\pi} n^{n+x+1/2} e^{-n}}{x(x+1) \cdots (x+n)} \exp \left\{ \sum_{j=1}^{\nu-1} \frac{B_{j+1}(-x)}{j(j+1)n^j} + O\left(\frac{1}{n^\nu}\right) \right\}.$$

Theorem 3. As $x \rightarrow \infty$, uniformly with respect to arbitrary $\theta = O(\sqrt{x})$ and arbitrary $a = O(1)$, the expansions

$$\Gamma(1+x) = \sqrt{2\pi} (x+\theta)^{x+1/2} \exp \left\{ -(x+\theta) + \frac{y^2}{2} + \sum_{j=1}^{\nu-1} \frac{v_j(y, a)}{(\sqrt{x+\theta})^j} + O\left(\frac{1}{(\sqrt{x+\theta})^\nu}\right) \right\},$$

$$\Gamma(1+x) = \sqrt{2\pi} (x+\theta)^{x+1/2} e^{-(x+\theta)-y^2/2} \left\{ 1 + \sum_{j=1}^{\nu-1} \frac{w_j(y, a)}{(\sqrt{x+\theta})^j} + O\left(\frac{1}{(\sqrt{x+\theta})^\nu}\right) \right\},$$

$$\frac{1}{\Gamma(1+x)} = \frac{e^{x+\theta-y^2/2}}{\sqrt{2\pi} (x+\theta)^{x+1/2}} \left\{ 1 + \sum_{j=1}^{\nu-1} (-i)^j \frac{w_j(iy, 1-a)}{(\sqrt{x+\theta})^j} + O\left(\frac{1}{(\sqrt{x+\theta})^\nu}\right) \right\},$$

hold, where

$$y = \frac{\theta - a}{\sqrt{x+\theta}},$$

$$v_j(y, a) = \sum_{k=0}^{1+[j/2]} \frac{B_k(a) C_{j-k+2}^k y^{j-2k+2}}{(j-k+1)(j-k+2)},$$

and $w_j(y, a)$ are polynomials of degree $3j$ in y of parity j , satisfying the recurrence relations

$$w_j(y, a) = \sum_{k=0}^{j-1} \left(1 - \frac{k}{j}\right) w_k(y, a) v_{j-k}(y, a), \quad w_0(y, a) = 1.$$

(The imaginary unit i enters the expression $(-i)^j w_j(iy, 1 - a)$ only in even powers.)

Theorems 2 and 3 are direct consequences of Theorem 1. All the assertions formulated can also be given in a complex version. Thus, Theorem 2 remains valid if x is replaced by a complex variable z as $|z| \rightarrow \infty$, uniformly with respect to an arbitrary complex quantity $\theta = O(1)$ and with respect to $|\arg z| \leq \pi - \varepsilon$. Theorem 3 holds when x is replaced by z , uniformly with respect to arbitrary complex quantities $\theta = O(\sqrt[3]{z})$, $a = O(1)$, and with respect to $|\arg z| \leq \pi - \varepsilon$. This makes it possible, in particular, to describe the asymptotic behavior of $\Gamma(1+z)$ for $z = x + iy$. For example, one may state the following result:

Theorem 4. *For $z = x \pm iy$, $y \rightarrow \infty$, the following equalities hold uniformly with respect to $x = O(1)$:*

$$\begin{aligned} \Gamma(1+z) &= \sqrt{2\pi} y^{x+1/2} \exp \left\{ -\frac{\pi}{2} y \pm i \left[\frac{\pi}{2} \left(x + \frac{1}{2} \right) + y(\ln y - 1) \right] + \sum_{j=1}^{\nu-1} \frac{(\mp i)^j B_{j+1}(-x)}{j(j+1)y^j} + O\left(\frac{1}{y^\nu}\right) \right\}, \\ \Gamma(1+z) &= \sqrt{2\pi} y^{x+1/2} \exp \left\{ -\frac{\pi}{2} y \pm i \left[\frac{\pi}{2} \left(x + \frac{1}{2} \right) + y(\ln y - 1) \right] \right\} \times \\ &\quad \times \left\{ 1 + \sum_{j=1}^{\nu-1} (\mp i)^j \frac{b_j(-x)}{y^j} + O\left(\frac{1}{y^\nu}\right) \right\}. \end{aligned}$$

From the complex version of Theorem 3 one can determine the asymptotic behavior of $\Gamma(1+x \pm iy)$ as $y \rightarrow \infty$ and $x = O(\sqrt[3]{y})$. For example, we immediately obtain

$$|\Gamma(1+x \pm iy)| = \sqrt{2\pi} y^{x+1/2} e^{-\frac{\pi}{2} y} \left\{ 1 + O\left(\frac{1}{\sqrt[3]{y}}\right) \right\}$$

uniformly with respect to $x = O(\sqrt[3]{y})$.

We note that, instead of the expansions of $\Gamma(1+x)$ and $\Gamma(1+z)$, one may everywhere write, with some obvious changes, the expansions of $\Gamma(x)$ and $\Gamma(z)$.

In the last two theorems the behavior of a frequently occurring ratio is described.

Theorem 5. *As $|z| \rightarrow \infty$, uniformly with respect to an arbitrary complex $\theta = O(1)$ and with respect to $|\arg z| \leq \pi - \varepsilon$, the following expansions hold:*

$$\frac{\Gamma(z+\theta)}{\Gamma(z)} = z^\theta \exp \left\{ \sum_{j=1}^{\nu-1} (-)^j \frac{B_{j+1} - B_{j+1}(\theta)}{j(j+1)z^j} + O\left(\frac{1}{z^\nu}\right) \right\},$$

$$\frac{\Gamma(z+\theta)}{\Gamma(z)} = z^\theta \left\{ 1 + \sum_{j=1}^{\nu-1} \frac{C_\theta^{(j)}}{z^j} + O\left(\frac{1}{z^\nu}\right) \right\},$$

$$\frac{\Gamma(z)}{\Gamma(z+\theta)} = z^{-\theta} \left\{ 1 + \sum_{j=1}^{\nu-1} (-)^j \frac{C_{1-\theta}^{(j)}}{z^j} + O\left(\frac{1}{z^\nu}\right) \right\},$$

$$C_\theta^{(j)} = \theta(\theta-1)\cdots(\theta-j) \sum_{k=0}^{j-1} g_{jk}(\theta-j-1)\cdots(\theta-j-k),$$

$$g_{j0} = \frac{1}{j+1}, \quad g_{j,j-1} = \frac{1}{2^j j!}, \quad g_{jk} = \frac{(j+k)g_{j-1,k} + g_{j-1,k-1}}{j+k+1}, \quad k=1, \dots, j-2.$$

(The values of the polynomials $C_\theta^{(j)}$ for $\theta = j+1, j+2, \dots$ are the Stirling numbers of the first kind $C_\theta^{(j)}$, and for $\theta = -1, -2, \dots$ the Stirling numbers of the second kind $\mathfrak{S}_{-\theta}^{(j)}$.)

Theorem 6. As $|z| \rightarrow \infty$, uniformly with respect to an arbitrary complex $\theta = O(1)$ and with respect to $|\arg z| \leq \pi - \varepsilon$, the following expansions hold:

$$\frac{\Gamma(z+\theta\sqrt{z})}{\Gamma(z)} = z^{\theta\sqrt{z}} \exp \left\{ \frac{\theta^2}{2} + \sum_{j=1}^{\nu-1} (-)^j \frac{\alpha_j}{(\sqrt{z})^j} + O\left(\frac{1}{(\sqrt{z})^\nu}\right) \right\},$$

$$\frac{\Gamma(z+\theta\sqrt{z})}{\Gamma(z)} = z^{\theta\sqrt{z}} e^{\theta^2/2} \left\{ 1 + \sum_{j=1}^{\nu-1} (-)^j \frac{\beta_j}{(\sqrt{z})^j} + O\left(\frac{1}{(\sqrt{z})^\nu}\right) \right\},$$

$$\frac{\Gamma(z)}{\Gamma(z+\theta\sqrt{z})} = z^{-\theta\sqrt{z}} e^{-\theta^2/2} \left\{ 1 + \sum_{j=1}^{\nu-1} (-)^j \frac{\gamma_j}{(\sqrt{z})^j} + O\left(\frac{1}{(\sqrt{z})^\nu}\right) \right\},$$

where

$$\alpha_j = \sum_{k=0}^{[(j+1)/2]} (-)^k \frac{C_j^k B_k \theta^{j-2k+2}}{(j-k+1)(j-k+2)},$$

$$\beta_j = \sum_{k=0}^{j-1} \left(1 - \frac{k}{j}\right) \beta_k \alpha_{j-k}, \quad \beta_0 = 1,$$

$$\gamma_j = - \sum_{k=0}^{j-1} \left(1 - \frac{k}{j}\right) \gamma_k \alpha_{j-k}, \quad \gamma_0 = 1.$$

Another analogue of Stirling' s formula is contained in work (2).

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
2 X 1967

REFERENCES

1. E. Copson, *Asymptotic Expansions*, Moscow, 1966.
2. V. M. Kalinin, *Theory of Probability and Its Applications*, **12**, no. 1, 24 (1967).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.