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Abstract

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MATHEMATICS

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ON A GENERALIZED FORMULA OF M. RIESZ

(Presented by Academician S. N. Bernstein on 4 III 1968)

1°. Denote by Π_n the set of all trigonometric polynomials of order $\leq n$. For a given polynomial

$$\Phi_n(t) = \sum_{k=0}^n r_k \sin(kt + \alpha_k)$$

construct the polynomial

$$\tilde{\Phi}_n(t) = r_n + 2 \sum_{k=0}^{n-1} r_k \cos[(n-k)t + \alpha_n - \alpha_k],$$

which we shall call associated with the polynomial Φ_n . In ⁽¹⁾ the following theorem was established:

If $t \in \Pi_n$, then the identity

$$\int_0^{2\pi} t(x + \theta) \Phi_n(\theta) d\theta = \frac{\pi}{2n} \sum_{r=1}^{2n} t(x + \psi_r) (-1)^{r-1} \tilde{\Phi}_n(\psi_r), \quad (1)$$

holds, where

$$\psi_r = \frac{2r-1}{2n} \pi - \frac{\alpha_n}{n}.$$

It is interesting that if

$$\Phi_n(t) = \frac{1}{\pi} \left(\frac{\sin(n+1/2)t}{2 \sin t/2} \right)',$$

then

$$\tilde{\Phi}_n(t) = \frac{\sin^2 nt/2}{\pi \sin^2 t/2}.$$

Therefore, with this choice of the polynomial Φ_n , equality (1) becomes the well-known identity of M. Riesz ⁽²⁾

$$t'(x) = \frac{1}{2n} \sum_{r=1}^{2n} t(x + \varphi_r) \frac{(-1)^{r-1}}{2 \sin^2 \varphi_r/2}, \quad t \in \Pi_n, \quad \varphi_r = \frac{(2r-1)\pi}{2n}. \quad (2)$$

The present note is devoted to extending identity (1) to the complex domain and to the case of semi-entire trigonometric polynomials.

2°. Let D be an arbitrary finite domain with simply connected complement D_1 and rectifiable boundary Γ . Denote by $w = \Phi(z)$ the function mapping D_1 conformally onto the domain $|w| \geq 1$ of the w -plane under the condition $\Phi(\infty) = \infty$. Let $z = \psi(w)$ be the inverse function. Denote by $A(D)$ the set of all functions $f(z)$ continuous in the closed domain $\bar{D}$ and analytic at every interior point of D . Put

$$\|f\| = \max_{z \in \bar{D}} |f(z)|.$$

Define the shift f_t of a function $f \in A(D)$ by the real number t , $-\infty < t < \infty$, by means of the equality

$$f_t(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f\{\psi[\Phi(\zeta)e^{it}]\}}{\zeta - z} d\zeta. \quad (3)$$

The transformation (3) has been well studied in the works of V. K. Dzyadyk (3,4). For fixed $\Phi_n \in \Pi_n$ introduce the operators

$$\sigma(f) = \sigma(f, z) = \int_{-\pi}^{\pi} f_t(z) \Phi_n(t) dt, \quad f \in A(D),$$

$$U(f, z) = \frac{\pi}{2n} \sum_{r=1}^{2n} f_{\psi_r}(z) (-1)^{r-1} \widetilde{\Phi}_n(\psi_r).$$

Theorem 1. If f is a polynomial of degree $\leq n$, then the identity

$$\sigma(f) = U(f)$$

holds.

We outline the proof. Denote by F_n the Faber polynomial of degree n generated by the domain D . It is known that any polynomial of degree n is expressed linearly in terms of the polynomials $\{F_k\}$, $0 \leq k \leq n$. Therefore it suffices to prove the identity only for the polynomials F_k , $0 \leq k \leq n$.

The Faber polynomials have the important property that

$$(F_k)_t = F_k e^{ikt}.$$

Therefore

$$\sigma(F_k) = F_k \int_{-\pi}^{\pi} \Phi_n(t) e^{ikt} dt, \quad (4)$$

$$U(F_k) = F_k \frac{\pi}{2n} \sum_{r=1}^{2n} e^{ik\psi_r} (-1)^{r-1} \widetilde{\Phi}_n(\psi_r). \quad (5)$$

By virtue of identity (1), valid for any trigonometric polynomial of order n ,

$$\int_{-\pi}^{\pi} \Phi_n(t) e^{ikt} dt = \frac{\pi}{2n} \sum_{r=1}^{2n} e^{ik\psi_r} (-1)^{r-1} \tilde{\Phi}_n(\psi_r).$$

Therefore it follows from (4) and (5) that $\sigma(F_k) = U(F_k)$.

Corollary. If the kernel Φ_n is such that

$$\tilde{\Phi}_n(\psi_r) \geq 0, \quad r = 1, 2, \dots, 2n, \quad (6)$$

then for any polynomial f of degree $\leq n$ the inequality

$$\left\| \int_{-\pi}^{\pi} f_t(z) \Phi_n(t) dt \right\| \leq \pi r_n \max_{r=1,2,\dots,2n} \|f_{\psi_r}(z)\| \quad (7)$$

holds.

Indeed, by Theorem 1 and (6) we have

$$\left\| \int_{-\pi}^{\pi} f_t(z) \Phi_n(t) dt \right\| \leq \frac{\pi}{2n} \max_{r=1,2,\dots,2n} \|f_{\psi_r}(z)\| \sum_{r=1}^{2n} \tilde{\Phi}_n(\psi_r). \quad (8)$$

One can verify that the last sum is equal to $2nr_n$. Therefore (7) follows from (8).

Inequality (7) is a generalization of the classical inequality of S. N. Bernstein on the maximum modulus of the derivative of a trigonometric polynomial.

3°. As is known, an expression of the form

$$\tau_n(x) = \sum_{k=0}^n (a_k \cos(k + 1/2)x + b_k \sin(k + 1/2)x),$$

where $\{a_k\}_{k=0}^n, \{b_k\}_{k=0}^n$ are constants, is called a half-integer trigonometric polynomial of order n . We denote the set of all such polynomials by $\Pi_{n+1/2}$. It is obvious that any polynomial $\Phi \in \Pi_{n+1/2}$ can be written in the form

$$\Phi(x) = \sum_{k=0}^n r_k \sin((k + 1/2)x + \alpha_k), \quad (9)$$

where r_k and α_k are real numbers, with $r_k \geq 0, 0 \leq k \leq n$.

Let Φ be a fixed polynomial from $\Pi_{n+1/2}$. Associate with it the polynomial

$$\tilde{\Phi}(x) = 2 \sum_{k=0}^n r_k \cos[(n+1)x - (k+1/2)x + \alpha_n - \alpha_k],$$

which we shall call the polynomial associated with the polynomial (9).

Theorem 2. Let $t_n \in \Pi_{n+1/2}$; then the identity holds

$$\int_0^{2\pi} t_n(x+\theta)\Phi(\theta) d\theta = \frac{\pi}{2n+2} \sum_{r=1}^{2n+2} t_n(x+\delta_r)(-1)^{r-1}\tilde{\Phi}(\delta_r), \quad (10)$$

where

$$\delta_r = \frac{(2r-1)\pi}{2n+2} - \frac{\alpha_n}{n+1}.$$

Proof. Obviously, it is enough to prove equality (10) only for the functions $\{\cos(k+1/2)x\}_{k=0}^n$, $\{\sin(k+1/2)x\}_{k=0}^n$. Let us verify, for example, identity (10) for $c_k(x) = \cos(k+1/2)x$, $0 \leq k \leq n$. After simple calculations we obtain that

$$\int_0^{2\pi} c_k(x+\theta)\Phi(\theta) d\theta = -\pi r_k \sin[(k+1/2)x - \alpha_k]. \quad (11)$$

Denote the right-hand side of (10) by $U(t_n, x)$. Then, by virtue of (11), it is necessary to prove that

$$U(c_k, x) = -\pi r_k \sin[(k+1/2)x - \alpha_k].$$

This equality follows from the equalities

$$\sum_{r=1}^{2n+2} (-1)^{r-1} \cos p\delta_r = \sum_{r=1}^{2n+2} (-1)^{r-1} \sin p\delta_r = 0 \quad 0 \leq p < n+1;$$

$$\sum_{r=1}^{2n+2} (-1)^{r-1} \cos(n+1)\delta_r = (2n+2) \sin \alpha_n;$$

$$\sum_{r=1}^{2n+2} (-1)^{r-1} \sin(n+1)\delta_r = (2n+2) \cos \alpha_n.$$

Let us give some applications of Theorem 2. Let

$$\Phi(x) = \sum_{k=0}^n \frac{2k+1}{2} \sin(k+1/2)x. \quad (12)$$

It is easy to check that in this case

$$\tilde{\Phi}(x) = \frac{\sin^2(n+1)x/2}{\sin^2 x/2} \cos \frac{x}{2}, \quad \frac{1}{\pi} \int_0^{2\pi} t(x+\theta) \Phi(\theta) d\theta = t'(x) \quad (13)$$

for any $t \in \Pi_{n+1/2}$. Therefore, if one takes into account that $\sin^2 \frac{n+1}{2} \delta_r = 1/2$, then by Theorem 2 we obtain

Corollary. For any polynomial $t \in \Pi_{n+1/2}$ the identity holds

$$t'(x) = \frac{1}{4n+4} \sum_{r=1}^{2n+2} t(x+\varphi_r) (-1)^{r-1} \frac{\cos \varphi_r/2}{\sin^2 \varphi_r/2}, \quad \varphi_r = \frac{2r-1}{2n} \pi. \quad (14)$$

Remark. Identity (14) can also be derived from M. Riesz' s identity (2). Applying identity (14) to the case where $t_n(x) = \sin(n+1/2)x$, and putting $x=0$, we obtain

$$\sum_{r=1}^{2n+2} \operatorname{ctg}^2 \frac{2r-1}{4n+4} \pi = (2n+1)(2n+2).$$

Theorem 3. If the kernel Φ_n is such that

$$\tilde{\Phi}_n(\delta_r) \geq 0, \quad r = 1, 2, \dots, (2n+2), \quad (15)$$

then for any $t \in \Pi_{n+1/2}$ the inequality holds

$$\left\| \int_0^{2\pi} t(x+\theta) \Phi_n(\theta) d\theta \right\| \leq \left(\frac{\pi}{n+1} \sum_{j=0}^n r_j \frac{\sin[\alpha_n(n-j+1/2)/(n+1) + \alpha_j - \alpha_n]}{\sin[(n-j+1/2)\pi/(2n+2)]} \right) \|t\|. \quad (16)$$

Inequality (16) is sharp in the sense that for

$$\Phi_n(\theta) = \sum_{k=0}^n \cos(k+1/2)\theta \quad (17)$$

for every $t \in \Pi_{n+1/2}$ equality holds in (16).

We outline the proof. From Theorem 2 it follows that for every $t \in \Pi_{n+1/2}$

$$\left\| \int_0^{2\pi} t(x + \theta) \Phi_n(\theta) d\theta \right\| \leq \left(\frac{\pi}{2n+2} \sum_{r=1}^{2n+2} |\tilde{\Phi}_n(\delta_r)| \right) \|t\|. \quad (18)$$

It can be verified that if inequalities (15) are satisfied, then

$$\sum_{r=1}^{2n+2} |\tilde{\Phi}_n(\delta_r)| = 2 \sum_{j=0}^n r_j \frac{\sin[\alpha_n(n-j+1/2)/(n+1) + \alpha_j - \alpha_n]}{\sin[(n-j+1/2)\pi/(2n+2)]}.$$

Therefore (16) follows from (18). Let Φ_n be defined according to (17). By simple calculations we obtain that in this case

$$\tilde{\Phi}_n(x) = \frac{\sin^2(n+1/2)t}{\sin t/2}. \quad (19)$$

Thus, inequalities (15) are satisfied. Hence Theorem 3 is applicable. But in fact in this case equality holds in (16), since it is directly seen that the right-hand side of (16) is equal to $\pi\|t\|$. It can be verified that if Φ_n is defined according to (17), then for every $t \in \Pi_{n+1/2}$

$$\int_0^{2\pi} t(x + \theta) \Phi_n(\theta) d\theta = \pi t(x).$$

Therefore the left-hand side of (16) is also equal to $\pi\|t\|$.

Theorem 4. If the kernel Φ_n is such that the inequalities

$$\Phi_n(\delta_r) \geq 0, \quad r = 1, 2, \dots, n, \quad (20)$$

$$\Phi_n(\delta_r) \leq 0, \quad r = n+1, n+2, \dots, 2n+2, \quad (21)$$

are satisfied, then for every $t \in \Pi_{n+1/2}$

$$\left\| \int_0^{2\pi} t(x + \theta) \Phi_n(\theta) d\theta \right\| \leq \left(\frac{2\pi}{n+1} \sum_{j=0}^n r_j \frac{(-1)^{n-j}}{\cos(2j+1)\pi/(2n+2)} \right) \|t\|.$$

From equality (13) it is seen that if Φ_n is defined according to (12), then inequalities (20), (21) are satisfied, and hence Theorem 4 is applicable in this case.

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