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Abstract

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MATHEMATICS

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ON MINIMA OF THE NORM FUNCTION IN IMAGINARY QUADRATIC FIELDS

(Presented by Academician Yu. V. Linnik on 21 III 1968)

Let $k = Q(\sqrt{-\Delta})$; $-\Delta$ be the discriminant of k , $\Delta > 0$; $h(\Delta)$ the number of ideal classes of k ; $\mathfrak{a}$ an integral ideal of k ; K_ν ($\nu = 1, \dots, h$) the ideal classes in k ; $a_\nu = \min N\mathfrak{a}$ for $\mathfrak{a} \in K_\nu$; $a(\Delta) = \max a_\nu$ ($\nu = 1, \dots, h$). As is known, a_ν is the first coefficient of the reduced form

$$Q_\nu(x, y) = a_\nu x^2 + b_\nu xy + c_\nu y^2,$$

where $\Delta = 4a_\nu c_\nu - b_\nu^2$ and $N\mathfrak{a} = Q_\nu(x, y)$ for $\mathfrak{a} \in K_\nu$. Yu. V. Linnik showed ⁽¹⁾, on the basis of ergodic considerations, that

$$A(x) = \sum_{a_\nu \leq x} 1 = \frac{6}{\pi^2} x L(1, \chi_1) (1 + o(1)), \quad (1)$$

where $\chi_1(n) = (-\Delta/n)$, $x = \lambda\sqrt{\Delta}$, $\Delta \rightarrow \infty$, and $\Delta^{-1/4+\varepsilon} \leq \lambda \leq 3^{-1/2}$.

Therefore there can exist only a finite number of values of Δ such that

$$a(\Delta) \leq \lambda\sqrt{\Delta},$$

if $\lambda < 3^{-1/2}$. However, it has not yet been possible to find an effective bound for such values of Δ , since relation (1) is proved with the aid of Siegel's ineffective theorem on the real zero of $L(s, \chi_1)$. A recently published result of Baker ⁽²⁾, however, makes it possible to obtain effective results concerning the behavior of the quantity $a(\Delta)$.

In this note the following is proved.

Theorem. For any fixed value $h = h(\Delta)$ there exist constants $\varkappa$ and Δ_0 , depending on h , such that for $\Delta \geq \Delta_0$ the function

$$a(\Delta) \geq (\lg \Delta)^{-\varkappa} \sqrt{\Delta}.$$

The constants $\varkappa$ and Δ_0 are effectively computable.

Proof. Let $\left(\frac{D}{\Delta}\right) = 1$, $D > 0$, $\chi(n)$ be an even character mod D ; $X(n) = \chi(n) \left(\frac{-\Delta}{n}\right)$. Then, arguing analogously to the way this is done in ⁽³⁾, we obtain for $\sigma > 1$

$$\begin{aligned} L(2s, \chi^2) &= \frac{1}{2} \sum_{\nu=1}^h \sum'_{(x,y)} \frac{\chi(Q_\nu(x, y))}{Q_\nu^s(x, y)} = \\ &= \zeta(2s) \prod_{p|D} \left(1 - \frac{1}{p^{2s}}\right) \sum_{\nu=1}^h \chi(a_\nu) a_\nu^{-s} + \sum_{\nu=1}^h a_\nu^{-s} \Phi_\nu \left(\frac{b_\nu}{2a_\nu}\right), \end{aligned} \quad (2)$$

where

$$\Phi_\nu(z) = \sum_{y=1}^{\infty} \sum_{x=-\infty}^{+\infty} \frac{\chi(Q_\nu(x, y))}{((x + zy)^2 + \Delta(4a_\nu)^{-1}y^2)^s}.$$

The function $\Phi_\nu(z)$ is periodic; its period is equal to D , and therefore it can be represented by a Fourier series

$$\Phi_\nu(z) = \sum_{r=-\infty}^{+\infty} A_r(s, \nu) \exp \frac{2\pi i}{D} r z,$$

where

$$A_r(s, \nu) = D^{-1} \int_0^D \Phi_\nu(t) \exp \frac{2\pi i}{D} r t dt \quad (r = 0, \pm 1, \pm 2, \dots).$$

Direct calculations show that

$$A_0(s, \nu) = D^{-1} \left(\frac{\Delta}{4}\right)^{1/2-s} I(s) \sum_{\nu=1}^h a_\nu^{s-1} \sum_{t=1}^D \sigma_\nu(t) \zeta\left(2s-1, \frac{t}{D}\right), \quad (3)$$

where

$$I(s) = \int_{-\infty}^{+\infty} (1+v^2)^{-s} dv, \quad \sigma_\nu(t) = \sum_{l=0}^{D-1} \chi(Q_\nu(l, t)).$$

Using the identity, as $s \rightarrow 1+0$,

$$\zeta\left(2s-1, \frac{t}{D}\right) = (2(s-1))^{-1} - \gamma + \frac{\lg D}{D} - \sum_{m=1}^{D-1} \rho^{mt} \lg(1 - \rho^{-m}) + O(1),$$

where γ is Euler's constant, ρ is a primitive root of the equation $x^D = 1$, we can pass to the limit $s \rightarrow 1$ in the identities (2) and (3):

$$L(1, \chi)L(1, X) = \pi^2 \alpha \sum_{\nu=1}^h \frac{\chi(a_\nu)}{a_\nu} + D^2 \pi \Delta^{-1/2} \sum_{\nu=1}^h \lg a_\nu \sum_{t=1}^D \sigma_\nu(t) - 2\pi D^{-2} \Delta^{-1/2} \sum_{\mu=1}^{D-1} \lg(1 - \rho^{-\mu}) \sum_{\nu=1}^h \sum_{t=1}^D \sigma_\nu(t) \rho^{t\mu} + O\left(\Delta^{-1/2} \exp\left(\frac{-\pi\sqrt{\Delta}}{a(\Delta)D}\right)\right), \quad (4)$$

where α is an algebraic number.

We shall now show that, by choosing χ , we can obtain the equalities

$$\sum_{t=1}^D \sigma_\nu(t) = 0 \quad (\nu = 1, \dots, h). \quad (5)$$

If Δ is odd, we put $D = 8$, and $\chi(n) = (8/n)$; a direct count justifies (5). If $\Delta \equiv 8$ or $12 \pmod{16}$, then among the first $[\lg h / \lg 2] + 2$ odd primes we choose q , which is prime to Δ ; this can be done by the well-known theorem of Gauss, since $2^{i-1} \leq h$, where i is the number of distinct prime divisors of Δ . Let χ be an odd complex character mod q ; then X will be an odd complex character mod $q\Delta$. For such characters there exist formulas (5), allowing us to transform the left-hand side of (4), and, using a remark of Chowla (6), we shall also satisfy (5). After simple transformations (4) becomes the inequality

$$\sum_{\nu=1}^{(q-1)/2} a_\nu \lg \sin \frac{\pi\nu}{q} + \beta\pi \ll \Delta^{-1/2} \exp \frac{-\pi\sqrt{\Delta}}{a(\Delta)q}, \quad (6)$$

where β, a_ν are algebraic numbers whose heights are $\ll \Delta^{c_1}$, with c_1 depending only on h ; moreover, for sufficiently large Δ , the number $\beta \neq 0$ (Lemma XV from (4)), and since π is multiplicatively independent of $\sin \pi\nu/q$ ($\nu = 1, \dots, (q-1)/2$), all the hypotheses of Baker's theorem (2) are satisfied for the left-hand side of (6), while the right-hand side contradicts (2) for sufficiently large Δ , if one puts $\chi = q + 2 + \varepsilon$.

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Note: Figure translations are in progress. See original paper for figures.

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