

COEFFICIENTS OF EXTREMAL POLYNOMIALS

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Abstract

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MATHEMATICS

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COEFFICIENTS OF EXTREMAL POLYNOMIALS

AND CHEBYSHEV ALTERNANTS

(Presented by Academician S. N. Bernstein on 6 X 1967)

In the proposed article, relations are established between the number of alternants of extremal polynomials ⁽¹⁾ and the magnitude of their coefficients. Some of such relations for the polynomials of P. L. Chebyshev were found by V. A. Markov ⁽²⁾ and generalized to Descartes and Chebyshev systems by S. N. Bernstein ⁽³⁾. In the present work the author uses the method of functionals of E. V. Voronovskaya ⁽¹⁾ and certain geometric considerations ⁽⁴⁾.

We shall also rely on the following result, obtained in the paper of E. V. Voronovskaya ⁽⁵⁾: a segment of the form $(\nu_i)_0^n$ with s parameters equal to 0, including ν_0 , and $n + 1 - s$ variable parameters, is served by all polynomials and only by polynomials with number of alternants $q \geq s$.

Theorem 1. *Let, in such a segment $(\nu_i)_0^n$, the variable parameters $(\nu_{k_i})_1^{n+1-s}$ be fixed and not all equal to 0. If*

$$Q_n(x) = \sum_{k=0}^n y_k x^k$$

is one of the extremal polynomials of such a segment, then it solves the following Chebyshev problem: among the set of polynomials $\{P_n(x)\}$ with prescribed coefficients $(y_{k_i})_1^{n+1-s}$, find the one least deviating from 0 on $[0, 1]$.

Indeed, the given $Q_n(x)$ has the required coefficients and its deviation $L = 1$. If there is a $Y_n(x)$ with the same coefficients $(y_{k_i})_1^{n+1-s}$ and deviation $L_1 < 1$, then still $Y_n(\bar{\nu}) = N$. On the other hand,

$$|Y_n(\bar{\nu})| = N \max_{[0,1]} |Y_n(x)| < N,$$

which is contradictory.

Corollary 1. Since $Q_n(x)$ has in its distribution not fewer than s alternants, it serves, at some point, also a segment of the form $(\mu_k)_0^n = 0_0, \dots, 0_{s-1}, \mu_s, \dots, \mu_n$ and, consequently, solves the basic Chebyshev problem: among $\{P_n(x)\}$ with coefficients $(y_k)_s^n$, find the one least deviating from 0 on $[0, 1]$.

Corollary 2. Since the last problem (by the results of P. L. Chebyshev) has a unique solution, the problem formulated in Theorem 1 also has a unique solution (this also follows from Haar's theorem). Thus, the reduced polynomial with $q = s$ is uniquely determined by its $n + 1 - s$ coefficients, except for the constant term, in the whole set M of reduced polynomials of degree not higher than n .

We shall consider every

$$P(x) = \sum_{k=0}^n y_k x^k$$

as a point of the $(n + 1)$ -dimensional space $P(y_0, \dots, y_n)$. The set $\{P\}$ under the condition

$$\max_{[0,1]} |P| \leq 1$$

is closed, convex, and symmetric with respect to the origin; its boundary M consists of all reduced polynomials. Choose arbitrarily l coordinate axes $y_{k_1}, \dots, y_{k_l}$ and consider $\{p(y_{k_i})_{i=1}^l\}$ —the orthogonal projection of the points $\{P\}$ onto this l -dimensional subspace. We obtain a closed convex set, symmetric with respect to the origin; each of its

a point for $l < n + 1$ is the projection of some point of the set M . In what follows we put $0 < k_1 < k_2 < \dots < k_l$.

Theorem 2. The boundary of $\{p(y_{k_i})_{i=1}^l\}$ consists of all and only the projections of polynomials for which $q \geq n + 1 - l$.

1. Let $P_0 \in M$ and let $q \geq n + 1 - l$ for it; let $p_0(y_{k_i})_{i=1}^l$ be its projection. According to (5), there is an interval of the form $(v_i)_0^n$ (see Theorem 1) which, for fixed parameters $(v_{k_i}^{(0)})_{i=1}^l$, is served by the polynomial

$$P_0(x) = \sum_{k=0}^n y_k^{(0)} x^k.$$

That is,

$$\sum_{i=1}^l y_{k_i}^{(0)} v_{k_i}^{(0)} = N = \max_{P \in M} \sum_{i=1}^l v_{k_i}^{(0)} y_{k_i} = \max_{p \in \{p\}} \sum_{i=1}^l v_{k_i}^{(0)} y_{k_i}.$$

But a linear function on a closed bounded set attains its greatest value at its boundary point; consequently, p_0 is on the boundary of $\{p(y_{k_i})_{i=1}^l\}$.

2. Let p_0 be a boundary point; through it passes a supporting hyperplane to $\{p(y_{k_i})_{i=1}^l\}$; let its equation be of the form $\sum_{i=1}^l v_{k_i}^{(0)} y_{k_i} = C$. But the problem of drawing a supporting hyperplane of a given direction to a convex set in the case under consideration is equivalent to finding $\max |C|$ on this set; consequently:

$$\left| \sum_{i=1}^l v_{k_i}^{(0)} y_{k_i}^{(0)} \right| = \max_{p \in \{p\}} \sum_{i=1}^l v_{k_i}^{(0)} y_{k_i} = \max_{P \in M} \sum_{i=1}^l v_{k_i} y_{k_i} = N.$$

(If $\sum_{i=1}^l v_{k_i}^{(0)} y_{k_i}^{(0)} < 0$, put $-v_{k_i}^{(0)} = v_{k_i}^{(1)}$.) Thus every point p_0 is the projection of an extremal polynomial of an interval of the form $(v_i)_0^n$ and, according to (5), this polynomial has $q \geq n + 1 - l$.

Theorem 2 is valid for any choice of axes $(y_{k_i})_1^l$, if l is fixed and $k_i > 0$. Therefore, in what follows the projection set will be denoted by $\{p\}_l$.

Remark. As a consequence of the theorem we obtain that every convex function of l coefficients of the polynomial attains on M its greatest value, and every concave one its least value, on polynomials with $q \geq n + 1 - l$. A similar property was established by S. N. Bernstein ⁽³⁾ for oscillating polynomials.

Let $P_0(x)$ be a chosen normalized polynomial. Mark its l ($l < n$) coefficients $(y_{k_i}^{(0)})_{i=1}^l$ and consider the set $L_{k_i} \subset M$ of all normalized polynomials with the same l coefficients.

Theorem 3. *The set L_{k_i} contains either 1) a unique polynomial, or 2) an infinite set; moreover, in both cases there is in L_{k_i} a polynomial with $q \geq n - l$.*

Choose $0 < m \neq k_i$ (for lack of space we omit the case $m = 0$, which requires special consideration). Adjoin to the axes $(y_{k_i})_1^l$ the axis y_m , and let $\{p\}_{l+1}$ be the projection of $\{P\}$ into this $(l + 1)$ -dimensional subspace. In $\{p\}_{l+1}$ all projection points of the polynomials from L_{k_i} are common points of $\{p\}_{l+1}$ and of the straight line parallel to the axis y_m . By virtue of the convexity of $\{p\}_{l+1}$, they form a segment of a straight line (case 2), in the limit degenerating into a point (case 1). Its endpoints are boundary points of $\{p\}_{l+1}$, i.e., by Theorem 2, projections of polynomials with $q \geq n - l$.

Corollary. In case 2, the extremum of each coefficient with an index different from $\{k_i\}_1^l$ is attained in L_{k_i} on polynomials with $q \geq n - l$. Thus, the absolute values of all coefficients of arbitrary polynomials from L_{k_i} are majorized by the modulus of the corresponding coefficient of some polynomial

$$\sum_{k=0}^n \lambda_k x^k$$

with $q \geq n - l$, i.e. $|y_m| \leq |\lambda_m|$ (the inequality is strict for $0 < m \neq k_i$).

Theorem 4. The set L_{k_i} consists of the single polynomial $P_0(x)$ if and only if $q > n - l$ (case 1 of Theorem 3).

- 1) Let $q = n - l + k$ ($k \geq 1$). According to Corollary 2 of Theorem 1, $P_0(x)$ is the unique polynomial in M having $l - k + 1$ ($\leq l$) prescribed coefficients. 2) Let L_{k_i} consist of the single $P_0(x)$; then the straight line passing through the point P_0 parallel to the axis y_m has with L_{k_i} a single common point with projection on the boundary $\{p\}_{l+1}$. But any straight line parallel to the axis y_m and having common points with $\{p\}_{l+1}$ projects these points into the l -dimensional subspace formed by the axes $y_{k_1}, \dots, y_{k_l}$. Consequently, in $\{p\}_l$ the projection of P_0 is also a boundary point; by Theorem 2, $q \geq n - l + 1$.

Theorem 5. If L_{k_i} contains an infinite set of polynomials, then it contains exactly two with $q = n - l$ (case 2 of Theorem 3).

Indeed, by Theorem 4, for all of L_{k_i} we have $q \leq n - l$, and, by Theorem 3, in L_{k_i} there are polynomials with $q = n - l$, and, according to the corollary of Theorem 3, at least two such polynomials: these are the polynomials $Q_n^{(1)}(x)$ and $Q_n^{(2)}(x)$, which have, at x^m ($0 < m \neq k_i$), respectively the least $y_m^{(\min)}$ and the greatest $y_m^{(\max)}$ coefficients. According to Corollary 2 of Theorem 1, $y_m^{(\min)} < y_m^{(\max)}$. Let us prove that y_m cannot take intermediate values on polynomials with $q = n - l$ from L_{k_i} . Suppose there exists such a $Q_n^{(0)}(x)$ with $y_m^{(0)}$, where $y_m^{(\min)} < y_m^{(0)} < y_m^{(\max)}$. Construct a segment-functional $F = (\nu_i)_0^n$, served by $Q_n^{(0)}(x)$, in which all parameters $(\nu_{k_i}^{(0)})_1^l$ and $\nu_m^{(0)}$ are not equal to 0, and all the remaining ones are zero. This is always possible ⁽⁵⁾. Then:

$$F(Q_n^{(0)}) = \sum_{i=1}^l \nu_{k_i}^{(0)} y_{k_i}^{(0)} + \nu_m^{(0)} y_m^{(0)} = N \quad (\text{the norm of } F).$$

If $\nu_m^{(0)} > 0$, then $F(Q_n^{(2)}) > N$; if $\nu_m^{(0)} < 0$, then $F(Q_n^{(1)}) > N$, and neither is possible.

Thus, if in L_{k_i} there are more than two distinct polynomials with $q = n - l$, then, in addition to the prescribed ones, they also have other coinciding coefficients, which contradicts Corollary 2 of Theorem 1.

Theorem 6. If L_k contains an infinite set of polynomials, then among them there is also an infinite set with $q = n - l - 1$.

Adjoin to the axes $y_{k_1}, \dots, y_{k_l}, y_m$ ($m > 0$) one more axis and construct $\{p\}_{l+2}$. According to Theorem 2, for boundary points we have $q \geq n - l - 1$. Fixing $y_{k_1}^{(0)}, \dots, y_{k_l}^{(0)}$, we obtain a two-dimensional plane intersecting $\{p\}_{l+2}$ in a convex domain bounded by a continuous curve. All points of this curve are projections of polynomials with $q = n - l - 1$, except for two points with extremal coefficients at x^m , to which correspond polynomials with $q = n - l$.

Denote by M_s the set of all reduced polynomials of degree not higher than n with $q = s (\leq n)$. From the theorems proved, the following assertions follow:

1. If $P(x) \in M_s$, then: a) in M_{s+k} ($k > 0$) there is not a single polynomial with the same $n - s$ coefficients (except for the constant term); b) in M_s there is, besides $P(x)$, exactly one more; c) in M_{s-k} there are infinitely many of them.

Part a) follows from Theorem 4, part b) from Theorem 5, and part c) from Theorem 6.

2. If $P(x) \in M_s$ and any $l < n - s$ of its coefficients, except for the constant term, are prescribed, then: a) in M_{n-l+k} there is not a single polynomial from L_k ; b) in M_{n-l} there are exactly 2; c) in M_{n-l-k} there are infinitely many.
3. If $P(x) \in M_s$ and any $l > n - s$ of its coefficients, except for the constant term, are prescribed, then L_k consists of the single $P(x)$.

Assertion 2 is proved analogously to Assertion 1. Assertion 3 is obvious.

All the results obtained above do not depend on the class of intervals and polynomials. They may be extended to polynomials with respect to systems of functions of Descartes. This will form the subject of another paper.

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Note: Figure translations are in progress. See original paper for figures.

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