

SOME EXTREMAL PROBLEMS

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Abstract

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MATHEMATICS

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SOME EXTREMAL PROBLEMS

FOR UNIVALENT ANALYTIC FUNCTIONS

(Presented by Academician M. A. Lavrent'ev, 15 II 1968)

In this note we consider extremal problems for certain classes of analytic functions, connected with problems in the theory of quasiconformal mappings.

§ 1. Let, in the class S of univalent analytic functions in the disk $U : |z| < 1$,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

the problem be considered of finding the extremum of the real functional

$$I(f) = I[f(\zeta), \overline{f(\zeta)}, f'(\zeta), \overline{f'(\zeta)}, \dots, f^{(m)}(\zeta), \overline{f^{(m)}(\zeta)}] \quad (1)$$

for fixed $\zeta \in U$, where I is a continuously differentiable function of the variables $u_l, v_l, f^{(l)}(\zeta) = u_l + iv_l, l = 0, 1, \dots, m$, and

$$\sum_{l=0}^m |\partial I / \partial f^{(l)}| > 0.$$

By applying the variational method of M. Schiffer-G. M. Goluzin^(3,7), it is established that every function $w = f_0(z)$ extremal for $I(f)$ maps the disk U onto a domain $B_0 = f_0(U)$ without exterior points, which is the w -plane cut along a finite number of analytic arcs (of which at least one goes to ∞). Denote by Σ the set of all extremal functions. We shall consider here the case when the boundary of the domain B_0 consists of one arc γ with endpoints $b \neq \infty$ and ∞ . Then the function $f_0(z)$ is analytic everywhere on the circle $|z| = 1$, except at the point $z_0 = f_0^{-1}(\infty)$, at which $f_0(z)$ has an algebraic singularity. By virtue of univalence, in a neighborhood of the point $z = z_0$ the function $f_0(z)$ has an expansion of the form

$$f_0(z) = (z - z_0)^{-2} \sum_{s=0}^{\infty} d_s (z - z_0)^s, \quad d_0 \neq 0.$$

Consider the function $\omega = g_0(z) = \sqrt{f_0(z) - b}$, $b = f_0(z_1)$, where z_1 is the point of the circle $|z| = 1$ corresponding to the endpoint b of the boundary slit, at which $f'_0(z)$ has a simple zero, and a fixed branch of the root is taken. Using the smoothness of the function $g_0(z)$ on the circle $|z| = 1$, it is not hard to construct its quasiconformal continuation into the domain $U_1 : |z| > 1$, i.e., a quasiconformal automorphism $\omega = G(z)$ of the extended z -plane, which for $|z| > 1$ satisfies the Beltrami equation $\omega_z = \mu(z)\omega_{\bar{z}}$, $\|\mu\|_{L_\infty(U_1)} \equiv k(f) < 1$, and for $|z| \leq 1$ coincides with $g_0(z)$. Returning to the function $f_0(z)$, we obtain a quasiconformal homeomorphism of the z -plane onto the two-sheeted covering of the w -plane, glued along γ , coinciding with $f_0(z)$ for $|z| \leq 1$.

Introduce the class $O(g_0)$ of quasiconformal automorphisms $\omega = f(z)$ of the extended complex z -plane satisfying the conditions: 1) the mappings f are conformal in the disk $|z| < 1$; 2) $k(f) \leq q < 1$; 3) $f(0) = g_0(0)$, $f'(0) = g'_0(0)$, $f^{(p)}(\zeta) = g_0^{(p)}(\zeta)$, $p = 0, 1, \dots, m$; 4) $f(z_1) = 0$, $f(z_0) = \infty$.

For some $q < 1$ the class $Q(g_0)$ is nonempty. Consider the problem of finding $\inf k(f) = k_0$ in the class $Q(g_0)$.

We shall also consider the problem of finding $\inf k(f) = k_\beta$ in the class $Q_\beta(g_0)$ of quasiconformal homeomorphisms $\omega = f(z)$ of the extended complex z -plane, slit along a fixed arc $\beta : \theta' < \theta < \theta''$ (where $\arg z_0 < \theta' < \theta'' < \arg z_1$) of the circle $|z| = 1$, satisfying conditions 1)–4).

Lemma 1. *The class $Q_\beta(g_0)$ is compact.*

With the aid of appropriate variations of quasiconformal mappings^(5,6) it is established that the mapping $\omega = f_\beta(z)$, extremal in the class $Q_\beta(g_0)$ (i.e., such that $k(f_\beta) = k_\beta$), maps the z -plane with a slit along the arc β onto a domain D_β having no exterior points, and moreover the plane measure of the boundary of the domain D_β is zero.

Theorem 1. *If, for some nondegenerate arc β , the function $f_1(z) = f_\beta^2(z) + b$ belongs to the set Σ for $|z| < 1$, then $k_\beta = k_0 = 0$, and the function $\omega = f_1(z)$ maps the disk $|z| < 1$ onto the ω -plane with a rectilinear slit γ_1 .*

The proof of this theorem, which relates the extremal functions of the problems under consideration, rests on the following lemma, analogous to the corresponding results of^(5,6).

Lemma 2. *Let the quasiconformal mapping $\omega = f_\beta(z)$ be extremal in the class $Q_\beta(g_0)$. Then either f_β is an analytic function, or (for $k_\beta > 0$) there exists a rational function*

$$\psi(\omega) = \sum_{s=0}^m \frac{A_s}{\omega(\omega-a)(\omega-\omega_0)^{s+1}} + \sum_{j=1}^2 \frac{B_j}{\omega(\omega-a)^{j+1}} \neq 0, \quad (2)$$

where $A_s = \text{const}$, $B_j = \text{const}$, $\omega_0 = g_0(\xi)$, $a = g_0(0)$, such that the characteristic $\mu_\beta(\omega)$ of the inverse mapping $z = f_\beta^{-1}(\omega)$ at the points $\omega = f_\beta(U_1)$ is equal to $\mu_\beta(\omega) = k_\beta \psi(\omega)/|\psi(\omega)|$.

The function $\psi(\omega)$ is determined up to a constant positive factor.

It is proved that, under the hypotheses of Theorem 1, the second case in the assertion of Lemma 2 cannot occur and $k_\beta = k_0 = 0$. For this one uses the fact that the boundary of the domain D_β contains (in the set-theoretic sense) an analytic arc Γ_β corresponding to the arc β of the circle $|z| = 1$. Applying the special variation of the ω -plane constructed in ⁽²⁾, No. 3, we obtain that, under the assumption $k_\beta > 0$, on the arc Γ_β the inequality $\psi(\omega)d\omega^2 > 0$ must hold, and this contradicts (2).

§ 2. Let in the complex z -plane there be singled out $n \geq 1$ distinct finitely connected domains $D_1, D_2, \dots, D_n$, bounded by nondegenerate Jordan curves, and suppose that $\overline{D}_i \cap \overline{D}_j = \emptyset$ for $i \neq j$. Put $D = \bigcup_{j=1}^n D_j$. Denote by $Q(D, q)$ the class of quasiconformal homeomorphisms $w = f(z)$, $f(\infty) = \infty$, of the z -plane, conformal in the domains $D_1, \dots, D_n$, with dilatation $K(f) \leq q < \infty$ ($K(f) = [1 + k(f)]/[1 - k(f)]$), which carry prescribed finite points $z^1, z^2, \dots, z^m$, $m \geq 2$, not lying in D , into prescribed finite points $w^1, w^2, \dots, w^m$, respectively. Under the assumption that the class $Q(D, q)$ is nonempty, consider the following problem.

Problem $\tilde{A}$. Let $z_1, z_2, \dots, z_l$ be fixed points lying in D , and let $\alpha_1, \alpha_2, \dots, \alpha_l$ be nonnegative integers. In the class $Q(D, q)$, find the maximum of the real functional

$$F(f) = F(w_{0,1}, w_{1,1}, \dots, w_{\alpha_1,1}; w_{0,2}, w_{1,2}, \dots, w_{\alpha_2,2}; \dots; w_{0,l}, w_{1,l}, \dots, w_{\alpha_l,l}), \quad (3)$$

where $w_{p,j} = f^{(p)}(z_j) = u_{p,j} + iv_{p,j}$, $p = 0, 1, \dots, \alpha_j$; $j = 1, 2, \dots, l$, and

the function F is continuously differentiable with respect to $u_{p,j}$ and $v_{p,j}$, and

$$\sum_{j=1}^l \sum_{p=0}^{\alpha_j} |\partial F / \partial w_{p,j}| > 0.$$

We shall assume that $q > 1$ (for $q = 1$ the class $O(D, q)$ consists of a single linear mapping). Introduce the notation $\{\varphi(\xi)\}_{r,j} = \varphi^{(p)}(z_j)$.

Theorem 2. A quasiconformal homeomorphism $w = f_0(z)$, for which the maximum of $F(f)$ in the class $Q(D, q)$ is attained, has the following properties:

there exists a rational function $\psi_0(w)$, which may have simple poles and, moreover, only at the points $w^1, w^2, \dots, w^m$, and a constant χ , $0 \leq \chi \leq 2\pi$, such that the mapping $z = f_0^{-1}(w)$, inverse to the extremal one, has characteristic $\mu_0(w) = (f_0^{-1})_{\bar{w}} / (f_0^{-1})_w$, which is equal to zero for $w \in f_0(D)$, and at the points $w \notin f_0(D)$ has the form

$$|\mu_0(w)| = (q-1)/(q+1), \quad \arg \mu_0(w) = -\arg [e^{i\chi} \psi^*(w) + \psi_0(w)], \quad (4)$$

where

$$\psi^*(w) = \sum_{j=1}^l \sum_{p=0}^{\alpha_j} F_{w_{p,j}} \{ (f_0(\xi) - w)^{-1} \}_{p,j}. \quad (5)$$

In particular, if $F(f) = F(w_{0,1}, \dots, w_{0,l})$, then

$$\psi^*(w) = \sum_{j=1}^l F_{w_{0,j}} (w_{0,j} - w)^{-1}.$$

The proof of this theorem is carried out by the method of paper ⁽⁴⁾, using results analogous to Lemma 2.

Problem A can also be considered (analogously) for quasiconformal mappings of a torus on which global coordinates, varying in the complex plane, have been introduced. In this case the function F must be invariant with respect to $w_{0,1}, w_{0,2}, \dots, w_{0,l}$.

When concrete functionals and domains are considered, various distortion theorems are obtained for mappings of the classes under consideration. For example, for the class $Q(0, 1; U; 1 + \varepsilon)$ of quasiconformal automorphisms $w = f(z)$ of the complex z -plane with normalization $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$, conformal in the disk $U : |z| < 1$, with characteristics $\mu(z) = f_{\bar{z}}/f_z$, $\|\mu\|_{L_\infty(|z|>1)} \leq \varepsilon$, for small ε the following is valid.

Theorem 3. *If $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is the restriction to $|z| < 1$ of a mapping $w = f(z) \in Q(0, 1; U; 1 + \varepsilon)$, then, up to quantities of order ε^2 , the estimates*

$$1 - M\varepsilon \leq |a_1| \leq 1 + M\varepsilon, \quad M = 4\pi^{-1} \int_0^1 K(r) dr \simeq 2.3; \quad (6)$$

$$|a_n| \leq 2\varepsilon/(n-1), \quad n = 2, 3, \dots,$$

hold, where $K(r)$ is the complete elliptic integral of the first kind. The constants M and $2/(n-1)$ cannot be improved.

To obtain the estimates (6), the known variational formula is used

$$f(z) = z - \frac{z(z-1)}{\pi} \iint_{|\zeta|>1} \frac{\mu(\zeta) d\xi d\eta}{\zeta(\zeta-1)(\zeta-z)} + O(\varepsilon^2), \quad |z| \leq R < \infty, \quad \zeta = \xi + i\eta, \quad (7)$$

where the estimate of the remainder term depends on R . The form of the extremal mappings for which equalities are attained in (6) is established with the aid of a theorem of P. P. Belinskii (see ⁽¹⁾, Theorem 1).

In general, for small $\varepsilon = q - 1$, Theorem 2 makes it possible to obtain, for a broad class of functionals, estimates that are sharp in order with respect to ε . Indeed, if $F(\varphi) = 0$ for $\varphi(z) = z$, then in the class under consideration $\max F(f) = O(\varepsilon)$.

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