

# ON THE APPROXIMATION OF PERIODIC FUNCTIONS BY LINEAR METHODS OF APPROXIMATION

V. V. Zhuk

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**Abstract**

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**MATHEMATICS**

**V. V. ZHUK**

# ON THE APPROXIMATION OF PERIODIC FUNCTIONS BY LINEAR METHODS OF APPROXIMATION

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**1°.** In this note general theorems are given in which certain relations are established between functionals possessing norm properties and moduli of smoothness; a number of applications of these theorems are indicated to the problems: 1) determining structural properties of a function on the basis of the order of its approximation by linear methods of approximation; 2) finding the order of approximation of a function by linear methods of approximation, if the structural properties are prescribed by moduli of smoothness.

Among the numerous investigations devoted to these questions, we mention works related to the saturation problem, to direct and inverse theorems of the constructive theory of functions, and to estimates of the order of approximation by linear methods by means of moduli of smoothness.

The formulation of the saturation problem belongs to Favard. Its solution for a number of summability methods was first given by Zamansky <sup>(1)</sup>. F. I. Kharshiladze <sup>(2)</sup> and A. Kh. Turetskii <sup>(3)</sup> established certain general theorems. A survey of known results is contained in <sup>(3)</sup>. The first results relating to direct and inverse theorems are due to D. Jackson (see <sup>(4)</sup>, p. 114), S. N. Bernstein <sup>(5)</sup>, p. 8), Vallee-Poussin <sup>(6)</sup>. Further investigations are connected with the names of A. Zygmund <sup>(7)</sup>, S. M. Lozinskii <sup>(8)</sup>, S. B. Stechkin <sup>(9,10)</sup>, N. K. Bari <sup>(10)</sup>. A survey of known results is contained in <sup>(10)</sup>, see also <sup>(11)</sup>, Chs. 5 and 6. Among works relating to estimates of the order of approximation, we indicate <sup>(12-14)</sup>.

**2°. Notation and assumptions.**  $C_{2\pi}$  is the space of continuous  $2\pi$ -periodic functions with the usual normalization;  $f(x) \in C_{2\pi}$ ;  $\omega_k(\delta, f)$  ( $\omega_0(\delta, f) = \|f\|$ ) is its modulus of smoothness of order  $k$ ;  $E'_n(f)$  is the best approximation by trigonometric polynomials of order  $\leq n$ ;  $\sigma(f)$  is the Fourier series

$$\sigma(f) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx);$$

$F(x)$  is the function conjugate to an antiderivative of  $f(x) - a_0/2$ ; the numbers  $r > 0$ ,  $k \geq 0$  are integers;  $C$  ( $0 < C < \infty$ ) and  $M$  ( $0 \leq M < \infty$ ) are constants depending only on those arguments that will be written down;  $T_n(x)$  is a trigonometric polynomial of order not exceeding  $n$ ;  $\Phi(f)$  is a semiadditive, bounded ( $\|\Phi\| = \sup_f \|\Phi(f)\|/\|f\| < \infty$ ) functional, defined and nonnegative for all  $f(x) \in C_{2\pi}$ .

### 3°. Main theorems.

**Theorem 1.** If for any  $T_n(x)$

$$\left| \Phi(T_n) - M_1 n^{-l} \|T_n^{(l)}(x)\| \right| \leq M_2 n^{-(l+1)} \|T_n^{(l+1)}(x)\|,$$

then for any  $f(x)$

$$\left| \Phi(f) - M_1 \omega_k(n^{-1}, f) \right| \leq C_1(k) \left[ (\|\Phi\| + M_1 + M_2) \omega_{k+1}(n^{-1}, f) \right].$$

**Theorem 2.** If for every  $T_n(x)$

$$\left| \Phi(T_n) - M_1 n^{-r} \|T_n^{(r)}(x)\| \right| \leq M_2 n^{-(r+1)} \|\tilde{T}_n^{(r+1)}(x)\|,$$

then for every  $f(x)$

$$\left| \Phi(f) - M_1 \omega_r(n^{-1}, f) \right| \leq C_2(r) \left[ (\|\Phi\| + M_1) \omega_{r+1}(n^{-1}, f) + M_2 n \omega_{r+2}(n^{-1}, \tilde{F}) \right].$$

We note that  $n \omega_{r+2}(n^{-1}, \tilde{F}) \leq C_3(r) \omega_r(n^{-1}, f)$ .

**Theorem 3.** If for every  $T_n(x)$

$$\left| \Phi(T_n) - M_1 n^{-r} \|\tilde{T}_n^{(r)}(x)\| \right| \leq M_2 n^{-(r+1)} \|T_n^{(r+1)}(x)\|,$$

then for every  $f(x)$

$$\left| \Phi(f) - M_1 n \omega_{r+1}(n^{-1}, \tilde{F}) \right| \leq C_4(r) \left[ (\|\Phi\| + M_2) \omega_{r+1}(n^{-1}, f) + M_1 n \omega_{r+2}(n^{-1}, \tilde{F}) \right].$$

**Theorem 4.** If for every  $T_n(x)$

$$\left| \Phi(T_n) - M_1 n^{-k} \|\tilde{T}_n^{(k)}(x)\| \right| \leq M_2 n^{-(k+1)} \|\tilde{T}_n^{(k+1)}(x)\|,$$

then for every  $f(x)$

$$|\Phi(f) - M_1 n \omega_{k+1}(n^{-1}, \tilde{F})| \leq C_5(k) \left[ \|\Phi\| \tilde{E}_n(f) + n(M_1 + M_2) \omega_{k+2}(n^{-1}, \tilde{F}) \right].$$

**Theorem 5.** If the functional  $\Phi(f)$  has the properties: 1) for any constant  $C$ ,  $\Phi(C) = 0$ ; 2) for every  $T_n(x)$ ,  $M_1 \|T_n^{(r)}(x)\| \leq \Phi(T_n)$ , then for every  $f(x)$

$$\frac{M_1 n^r}{2} \left[ \omega_{r+1}(n^{-1}, f) + n \omega_{r+1}(n^{-1}, \tilde{F}) \right] \leq [2^{r+2} M_1 n^r + 11 \|\Phi\|] E_n(f) + \Phi(f).$$

A common feature of Theorems 1-4 is that the conditions are imposed on a narrow class of functions (in our case, on trigonometric polynomials), while the conclusion is made for all functions from the space  $C_{2\pi}$ . This idea, as applied to questions of approximation, has already been used earlier <sup>(12,13,17)</sup>, etc. The form of the conditions and the type of conclusions, apparently, are new. If in Theorems 1 and 4 one sets  $M_1 = 0$ , then general theorems are obtained in which the functional is estimated from above by means of moduli of smoothness and best approximations. Some special cases of these theorems are known <sup>(12,13)</sup>.

**4°. Some applications of the main theorems.** An important special case of the theorems given above is that in which the quantity  $\|U_n(f) - f(x)\|$  is taken as the functional  $\Phi(f)$ , where  $U_n(f)$  is a linear approximation process. We present some results belonging here.

**Theorem 6.** Let  $U_k(f)$  ( $k = 1, 2, \dots, s$ ) be linear operators from  $C_{2\pi}$  to  $C_{2\pi}$ , possessing the following properties for every  $T_n(x)$  ( $n = 0, 1, 2, \dots$ ): 1)  $U_k(T_n)$  is a trigonometric polynomial of order not exceeding  $n$ ; 2)  $U_k(\tilde{T}_n) = \tilde{U}(T_n)$ ; 3) for any natural  $p$ ,  $U_k^{(p)}(T_n) = U_k(T_n^{(p)})$ . Let the numbers  $p_k > 0$ , and  $m_k$  be integers and nonnegative,

$$r = \sum_{k=1}^s p_k m_k, \quad g = \sum_{k=1}^l m_k.$$

If for every infinitely differentiable function  $f(x)$

$$\|U_k(f)\| \leq M_k \|f^{(p_k)}(x)\| \quad (k = 1, 2, \dots, l),$$

$$\|U_k(f)\| \leq M_k \|\tilde{f}^{(p_k)}(x)\| \quad (k = l + 1, \dots, s),$$

then for every  $f(x) \in C_{2\pi}$  and for every natural  $n$

$$\left\| \prod_{k=1}^s (U_k)^{m_k}(f) \right\| \leq C_6(r) \left[ \prod_{k=1}^s \|U_k\|^{m_k} + \prod_{k=1}^s M_k^{m_k} n^r \right] \omega_r(n^{-1}, f),$$

if  $g$  is an even number, and

$$\left\| \prod_{k=1}^s (U_k)^{m_k}(f) \right\| \leq C_7(r) \left[ \prod_{k=1}^s \|U_k\|^{m_k} E_n(f) + \prod_{k=1}^s M_k^{m_k} n^{r+1} \omega_{r+1}(n^{-1}, \tilde{F}) \right],$$

if  $g$  is an odd number.

In connection with Theorem 6 we mention the papers of I. P. Natanson <sup>(15, 16)</sup>, in which questions are studied that pertain to the approximation of a function by the process

$$\{E - (E - I_n)^r\}(f),$$

where  $E$  is the identity operator from  $C_{2\pi}$  to  $C_{2\pi}$ , and  $I_n(f)$  is a singular integral whose kernel is subject to certain conditions, as well as the paper <sup>(18)</sup>, where this process is considered for the Jackson integral.

**Theorem 7.** Let  $U_n(f)$  be linear operators from  $C_{2\pi}$  to  $C_{2\pi}$ , for all  $n$ , possessing the following properties: 1) for any constant  $C$ ,  $U_n(C) = C$ ; 2)  $\|U_n\| \leq L$ ; 3) for any  $f(x)$ ,  $E_{An}(f) \leq C_8 \|U_n(f) - f\|$ , where  $A > 1$  is a natural number.

Then: 1) if for any  $T_n(x)$

$$M_1(n) \|T_n^{(r)}(x)\| \leq \|U_n(T_n) - T_n(x)\|,$$

then for any  $f(x)$

$$n^r M_1(An) \omega_r(n^{-1}, f) \leq C_9(r, A, C_8) [L + n^r M_1(An)] \sup_{m \geq n} \|U_m(f) - f(x)\|;$$

2) if for any  $T_n(x)$

$$M_1(n) \|\tilde{T}_n^{(r)}(x)\| \leq \|U_n(T_n) - T_n(x)\|,$$

then for any  $f(x)$

$$\begin{aligned} & n^r M_1(An) [\omega_{r+1}(n^{-1}, f) + n \omega_{r+1}(n^{-1}, \tilde{F})] \leq \\ & \leq C_{10}(r, A, C_8) [L + n^r M_1(An)] \sup_{m \geq n} \|U_m(f) - f(x)\|. \end{aligned}$$

We note that the first assertion of the theorem can be obtained from Theorem 2 of the paper <sup>(19)</sup>.

The results presented above make it possible, for many concrete methods of approximation, to obtain a complete description of the order of approximation

of functions in terms of moduli of smoothness. Here we shall indicate only one result of this kind.

Put

$$B_n(f) = 2^{-1} [S_n(x, f) + S_n(x + 2\pi(2n+1)^{-1})], \quad (1)$$

$$Z_n^{(l)}(f) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \left[ 1 - \left( \frac{k}{n} \right)^l \right] (a_k \cos kx + b_k \sin kx), \quad (2)$$

$$D_n(f) = \frac{3}{2\pi} \int_{-\infty}^{\infty} f\left(x + \frac{2t}{n}\right) \frac{\sin^4 t}{t^4} dt, \quad (3)$$

$$I_n(f) = \frac{3}{2\pi n(2n^2+1)} \int_{-\pi}^{\pi} f(x+t) \left[ \frac{\sin nt/2}{\sin t/2} \right]^4 dt, \quad (4)$$

$$\mathfrak{S}_n^{(p)}(f) = \frac{a_0}{2} + \sum_{k=1}^n (1 - C_{16} \sin^p kh_n) (a_k \cos kx + b_k \sin kx), \quad (5)$$

where it is assumed that  $h_n$  satisfies the condition

$$0 < K_1 \leq nh_n \leq \pi - K_1,$$

where  $K_1$  is a certain fixed number, and that, for all natural  $n$ ,

$$\|\mathfrak{S}_n^{(p)}\| \leq C_{11}.$$

The operators (1)–(4) bear in the literature, respectively, the names: the sums of S. N. Bernstein, the normal means of Zygmund, the Jackson–Vallée Poussin integral, and the Jackson integral. The operators (5) are a generalization of the well-known sums of V. Rogozinski and were considered earlier by G. I. Natanson, R. M. Trigub <sup>(14)</sup>, and the author <sup>(20)</sup>.

Let

$$X_n(f) = (E - Z_n^{(l)}) (E - \mathfrak{S}_n^{(p)}) (E - B_n)^k (E - D_n)^g (E - I_{n+2})^w(f),$$

where  $E$  is the identity transformation operator from  $C_{2\pi}$  into  $C_{2\pi}$ .

**Theorem 8.** Let the numbers  $l, p, g, w$  be integers and nonnegative,  $r = k + l + p + 2(g + w)$ . Then for every  $f(x)$ :

$$\omega_r(n^{-1}, f) \leq C_{12} \sup_{m \geq n} \|X_m(f)\| \leq C_{13} \omega_r(n^{-1}, f),$$

if  $p + l$  is an even number;

$$\begin{aligned} [\omega_{r+1}(n^{-1}, f) + n\omega_{r+1}(n^{-1}, \tilde{F})] &\leq C_{14} \sup_{m \geq n} \|X_m(f)\| \leq \\ &\leq C_{15} [\omega_{r+1}(n^{-1}, f) + n\omega_{r+1}(n^{-1}, \tilde{F})], \end{aligned}$$

if  $p + l$  is an odd number.

The constants  $C_{12}, C_{13}, C_{14}, C_{15}$  depend neither on  $n$  nor on  $f$ .

Some special cases of Theorem 8 are known (<sup>12-14,19</sup>; 5, p. 523).

**Remark 1.** If  $g^2 + w^2 = 0$ , then in the theorem the quantity  $\sup_{m \geq n} \|X_m(f)\|$  may be replaced by  $\|X_n(f)\|$ .

**Remark 2.** Let  $p^2 + k^2 = 0$ . Then for every  $f(x)$

$$\|X_n(f)\| = (3/2)^{w+g} \omega_r(n^{-1}, f) + O[\omega_{r+1}(n^{-1}, f) + n\omega_{r+2}(n^{-1}, \tilde{F})],$$

if  $l$  is an even number, and

$$\|X_n(f)\| = (3/2)^{w+g} n\omega_{r+1}(n^{-1}, \tilde{F}) + O[\omega_{r+1}(n^{-1}, f) + n\omega_{r+2}(n^{-1}, \tilde{F})],$$

if  $l$  is an odd number. Here  $O$  depends only on  $l, w, g$ .

**Remark 3.** Let  $w^2 + g^2 = 0$ . Then

$$\|X_n(f)\| = C_{16} n^p h_n^p(\pi/2)^k \omega_r(n^{-1}, f) + O(\omega_{r+1}(n^{-1}, f)),$$

if  $p + l$  is an even number, and

$$\|X_n(f)\| = C_{16} n^{p+1} h_n^p(\pi/2)^k \omega_{r+1}(n^{-1}, \tilde{F}) + O(E_n(f) + n\omega_{r+2}(n^{-1}, \tilde{F})),$$

if  $p + l$  is an odd number. Here  $O$  depends neither on  $n$  nor on  $f$ .

The results of the note are also valid for the space  $\mathcal{L}_{2\pi}$ .

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Novgorod Branch  
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Institute named after V. I. Ulyanov (Lenin)

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