

ON AN INTEGRAL EQUATION OF CONVOLUTION TYPE WITH $\backslash(N\backslash)$ KERNELS

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Abstract

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MATHEMATICS

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ON AN INTEGRAL EQUATION OF CONVOLUTION TYPE WITH N KERNELS

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The aim of this note is to investigate the integral equation of convolution type

$$\varphi(x) + \sum_{j=1}^N \int_{a_{j-1}}^{a_j} k_j(x-t)\varphi(t) dt = f(x), \quad -\infty < x < +\infty, \quad (1)$$

$$-\infty = a_0 < a_1 < a_2 < \dots < a_{N-1} < a_N = +\infty,$$

under the following assumptions: $k_j(x) \in L_1(-\infty, +\infty)$ ($j = 1, 2, \dots, N$), while the functions $f(x)$ and $\varphi(x)$ belong to one of the spaces $E(-\infty, +\infty) : L_p$ ($p \geq 1$), M , M^c , M^u , C , C^0 (see ^(4,5)).

In the case when $f(x)$ and $\varphi(x)$ belong to $L_2(-\infty, +\infty)$, under the assumption that $[1+K_j(x)]/[1+K_N(x)]$ ($j = 2, \dots, N-1$) are rational and $(1+K_1)(1+K_N)$ satisfies Hölder's condition, equation (1) was studied by I. B. Simonenko ⁽⁸⁾.

In this note it is proved that the homogeneous equation (1) has the same solutions in all the spaces from E . Some sufficient conditions for uniqueness of the solution are given, and a connection is established between the given equation and an integral equation with a Cauchy kernel, which reduces to an equivalent Riemann boundary-value problem for a system of $N-1$ pairs of functions. Cases are singled out in which the solution of equation (1) is obtained in closed form.

1. For equation (1) the following representation is valid

$$\begin{aligned} & \varphi(x) + \int_{-\infty}^{+\infty} \frac{k_1(x-t) + k_N(x-t)}{2} \varphi(t) dt + \\ & + \sum_{j=1}^{N-1} \int_{-\infty}^{+\infty} \frac{k_{j+1}(x-t) - k_j(x-t)}{2} \varphi(t) \operatorname{sgn}(t - a_j) dt = f(x). \end{aligned} \quad (2)$$

In the case $N = 2$, $a_1 = 0$ it was obtained by Yu. I. Cherskii ⁽¹⁰⁾.

The left-hand side of equation (2) defines in each of the spaces E a bounded linear operator $I + A$, where I is the identity operator. For the norm of the operator A the estimate

$$\begin{aligned} \|A\|_E &\leq \frac{1}{2} \left(\|k_1 + k_N\|_L + \sum_{j=1}^{N-1} \|k_{j+1} - k_j\|_L \right) \leq \\ &\leq \frac{1}{2} \left[\|k_1 + k_N\|_L + (N-1) \max_j \|k_{j+1} - k_j\|_L \right]. \end{aligned} \quad (3)$$

Represent the operator A as the sum of two operators

$$\begin{aligned} B\varphi &= \int_{-\infty}^0 k_1(x-t)\varphi(t) dt + \int_0^{\infty} k_N(x-t)\varphi(t) dt, \\ T\varphi &= \sum_{j=1}^{N-1} \int_{-\infty}^{+\infty} \frac{k_{j+1}(x-t) - k_j(x-t)}{2} \varphi(t) [\operatorname{sgn}(t - a_j) - \operatorname{sgn} t] dt. \end{aligned}$$

Lemma 1. The operator T is completely continuous in any of the spaces E and in the intersection (see (9)) of any two of these spaces.

It follows from ^(3,4) that the condition $1 + K_j(x) \neq 0$ ($j = 1, N$) is necessary and sufficient for the operator $I + A$ to be a Φ -operator in any of the spaces E , and also in the intersection of any two of these spaces. For the index of this Φ -operator the formula is valid

$$\chi = \operatorname{Arg} \frac{1 + K_1(x)}{1 + K_N(x)} \Bigg|_{-\infty}^{+\infty}.$$

Theorem 1. If $1 + K_j(x) \neq 0$ ($j = 1, N$), then in each of the spaces E the homogeneous equation (1) has the same solutions.

2. Considering henceforth equation (1) in the space L_1 and applying the Fourier transform to it, we obtain in the ring R the following singular equation with Cauchy kernel, equivalent to the given one:

$$\left[1 + \frac{K_1(x) + K_N(x)}{2} \right] \Phi(x) + \sum_{j=1}^{N-1} \left[\frac{K_{j+1}(x) - K_j(x)}{2} \frac{e^{ia_j x}}{\pi i} \int_{-\infty}^{+\infty} \frac{e^{ia_j t} \Phi(t)}{t - x} dt \right] = F(x). \quad (4)$$

The equation obtained, by introducing $(N-1)$ functions

$$\Psi_j(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{-ia_j t} \Phi(t)}{t - z} dt, \quad j = 1, 2, \dots, N-1,$$

can be reduced to an equivalent Riemann boundary-value problem for a system of $(N-1)$ pairs of functions in the ring $R_{(N-1 \times 1)}$ (see ^(1,2,6,7))

$$\begin{aligned} & [1 + K_{l+2}(x)]\Psi_{l+1}^+(x) + \sum_{j=l+2}^{N-1} [K_{j+1}(x) - K_j(x)]e^{i(a_j - a_{l+1})x}\Psi_j^+(x) \\ & = [1 + K_{l+1}(x)]\Psi_{l+1}^-(x) - \sum_{j=1}^l [K_{j+1}(x) - K_j(x)]e^{i(a_j - a_{l+1})x}\Psi_j^-(x) + e^{-ia_{l+1}x}F(x), \quad 0 \leq l \leq N-2; \end{aligned} \quad (5)$$

$$e^{ia_mx} [\Psi_m^+(x) - \Psi_m^-(x)] = e^{ia_{m+1}x} [\Psi_{m+1}^+(x) - \Psi_{m+1}^-(x)],$$

$$m = 1, 2, \dots, N-2.$$

In matrix form the boundary-value problem (5) is written as

$$\Psi^+(x) = G(x)\Psi^-(x) + F_1(x). \quad (6)$$

The elements $a_{pq} \in R$ of the matrix $G(x)$ are determined by the equalities

$$a_{pq} = \begin{cases} [K_q - K_{q+1}][1 + K_N]^{-1}e^{i(a_q - a_p)x}, & p \neq q, \\ 1 - [K_q - K_{q+1}][1 + K_N]^{-1}, & p = q. \end{cases}$$

Theorem 2. If $(N-1) \max \|k_{j+1} - k_j\|_L < 1/2$, then the integral equation (1) is unconditionally solvable in all spaces from E and has a unique solution.

The proof of the theorem can be obtained by reducing the boundary-value problem (6) to an equivalent operator equation in the ring $R_{(N-1 \times 1)}$.

Corollary. Let $(N-1) \max \|k_{j+1} - k_j\|_L = M_{N-1}$ and $1/2 \leq M_{N-1} < 2$. Then, if $\|k_1 + k_N\|_L < 2 - M_{N-1}$, the integral equation (1) is unconditionally solvable in all spaces from E and has a unique solution.

3. We shall assume that $1 + K_j(x) \neq 0$ ($j = 2, 3, \dots, N-1$). Suppose that at least one of the following $(N-1)$ conditions is satisfied: the functions $[1 + K_1(x)]/[1 + K_2(x)]$, $(1 + K_2(x))/(1 + K_3(x))$, ..., $[1 + K_l(x)]/[1 + K_{l+1}(x)] \in R^-$ and have no zeros in the lower half-plane, while $[1 + K_{l+2}(x)]/[1 + K_{l+3}(x)]$, $[1 + K_{l+3}(x)]/[1 + K_{l+4}(x)]$, ...,

$\dots, [1 + K_{N-1}(x)]/[1 + K_N(x)] \in R^+$ and have no zeros in the upper half-plane; $l = 0, 1, \dots, N - 2$.

Theorem 3. *Under each of the above $(N - 1)$ conditions, the boundary-value problem obtained for the system of $(N - 1)$ pairs of functions reduces to the successive solution of $(N - 1)$ Riemann boundary-value problems in the ring R , the first of which is solvable depending on the sign of the index χ , while the remaining $N - 2$ are unconditionally solvable.*

If $\chi \geq 0$, then the integral equation in any of the spaces E is unconditionally solvable and has χ linearly independent solutions. If $\chi < 0$, the homogeneous equation has only the trivial solution, and for solvability of the nonhomogeneous equation it is necessary and sufficient that $|\chi|$ solvability conditions be satisfied.

Proof. Under the stated assumptions, the first boundary condition in (5) may be regarded as a Riemann boundary-value problem in the ring R with index equal to χ ,

$$\Omega_{l+1}^+(x) = \frac{1 + K_{l+1}(x)}{1 + K_{l+2}(x)} \Omega_{l+1}^-(x) + \frac{e^{-ia_{l+1}x} F(x)}{1 + K_{l+2}(x)}, \quad (7)$$

where the notation has been introduced

$$\begin{aligned} \Omega_s^+(x) &= \Psi_s^+(x) + \sum_{j=s+1}^{N-1} \frac{K_{j+1}(x) - K_j(x)}{1 + K_{s+1}(x)} e^{i(a_j - a_s)x} \Psi_j^+(x) \in \dot{R}^+, \\ \Omega_s^-(x) &= \Psi_s^-(x) - \sum_{j=1}^l \frac{K_{j+1}(x) - K_j(x)}{1 + K_{l+1}(x)} e^{i(a_j - a_s)x} \Psi_j^-(x) \in \dot{R}^-, \end{aligned} \quad (8)$$

$$s = l + 1, \dots, N - 1.$$

After finding the functions $\Omega_s^\pm(x)$, using the relations

$$[\Psi_{s+1}^+(x) - \Psi_{s+1}^-(x)] e^{ia_{s+1}x} = [\Psi_s^+(x) - \Psi_s^-(x)] e^{ia_sx},$$

we shall each time obtain unconditionally solvable boundary-value problems for determining $\Omega_{s+1}^\pm(x)$:

$$\Omega_{s+1}^+(x) = \frac{1 + K_{s+1}(x)}{1 + K_{s+2}(x)} \Omega_{s+1}^-(x) + F_{s+1}(x),$$

where

$$F_{s+1}(x) = \frac{1 + K_{s+1}(x)}{1 + K_{s+2}(x)} e^{i(a_s - a_{s+1})x} [\Omega_s^+(x) - \Omega_s^-(x)].$$

Each of the obtained problems in the ring R has $\max(0, \chi)$ linearly independent solutions.

Taking into account that $\Omega_{N-1}^+(x) = \Psi_{N-1}^+(x)$, from $\Omega_{N-2}^+(x) \in \dot{R}^+$ we uniquely find $\Psi_{N-2}^+(x)$, and so on. Thus we find the functions $\Psi_{l+1}^+(x), \Psi_{l+2}^+(x), \dots, \Psi_{N-1}^+(x) \in \dot{R}^+$.

Using relation (8) and proceeding analogously, we find the functions $\Psi_1^+(x), \dots, \Psi_l^+(x)$ and $\Psi_1^-(x), \dots, \Psi_l^-(x)$. Then, by formula (8), we determine all the remaining functions $\Psi_{l+1}^-(x), \dots, \Psi_{N-1}^-(x)$.

From the relation $\Phi(x) = e^{ia_{jx}} [\Psi_j^+(x) - \Psi_j^-(x)]$, by the inverse Fourier transform we find the solutions of equation (1). The theorem is proved.

The method used above, with some modifications, extends to equation (1) under more general assumptions on the kernels.

Theorem 4. *Suppose that at least one of the $(N - 1)$ conditions is satisfied:*

$$1 + K_j(x) = (1 + S_1)[1 + M_j^-(x)]/[1 + T_j^-(x)],$$

where $j = 1, 2, \dots, l + 1$, $S_1(x) \in \dot{R}$, $M_j^-(x)$ and $T_j^-(x) \in R^-$, and

$$1 + K_j(x) = (1 + S_2)[1 + M_j^+(x)]/[1 + T_j^+(x)],$$

where $j = l + 2, \dots, N$, $S_2(x) \in \dot{R}$, $M_j^+(x)$ and $T_j^+(x) \in R^+$; $l = 0, 1, \dots, N - 2$. Under each of these $(N - 1)$ conditions, the solution of the integral equation (1) is obtained in closed form.

The method extends to the N -fold equation of convolution type

$$\varphi(x) + \int_{-\infty}^{+\infty} k_j(x - t)\varphi(t) dt = f(x), \quad a_{j-1} < x < a_j, \quad j = 1, 2, \dots, N.$$

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