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Abstract

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MATHEMATICS

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A PRIORI ESTIMATES OF SOLUTIONS OF ELLIPTIC EQUATIONS IN THE CLASS OF ANALYTIC FUNCTIONS

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Let u be the solution of the Dirichlet problem for an elliptic equation of second order with analytic coefficients in a domain $G \subset R^n$ with analytic boundary $\dot{G}$:

$$\sum_{|\beta| \leq m} a_\beta(x) D^\beta u = f; \quad x \in G; \quad u = \varphi; \quad x \in \dot{G}.$$

The main result of the paper is the establishment of estimates up to the boundary for the solution u and derivatives of the solution in the class of analytic functions. In obtaining these estimates we used the ideas of the work ⁽¹⁾.

§ 1. **Norms and their properties.** We shall denote by T_j first-order differential operators with analytic coefficients:

$$T_j = \sum_{i=1}^n b_{ij}(x) D_i, \quad 1 \leq j \leq s.$$

The operators T_j satisfy two assumptions:

1. On the boundary $\dot{G}$ the operators T_j are differentiation operators in directions tangent to the boundary.
2. At every point $x \in \dot{G}$, derivatives in tangential directions can be represented in the form of linear combinations of the T_j .

As such operators one may take the operators $a_j(x) D_i - a_i(x) D_j$, where the functions $a_j(x)$, analytic in G , are equal on the boundary $\dot{G}$ to $\cos(N, x_j)$ (N is the exterior normal to $\dot{G}$).

For functions $f \in C_{\infty, k+\alpha}(G \cup \dot{G})$, i.e. Hölder-continuous with exponent α , together with all their derivatives of the form

$$D^\beta f, (D^\beta T_{j_1} \dots T_{j_l})f; \quad |\beta| \leq k, \quad l = 1, 2, \dots,$$

we introduce

$$\begin{aligned} |f|_{0,k+\alpha} &= \|f\|_{C_{k+\alpha}(G)}, & |f|_{l,k+\alpha} &= \max_{j_\nu} \left\| \left(\prod_{\nu=1}^l T_{j_\nu} \right) f \right\|_{C_{k+\alpha}(G)}, \\ |\dot{f}|_{0,k+\alpha} &= \|f\|_{C_{k+\alpha}(\dot{G})}, & |\dot{f}|_{l,k+\alpha} &= \max_{j_\nu} \left\| \left(\prod_{\nu=1}^l T_{j_\nu} \right) f \right\|_{C_{k+\alpha}(\dot{G})}. \end{aligned}$$

The role of norms in $C_{\infty,k+\alpha}$ will be played by the formal power series

$$\|f\|_{\rho,k+\alpha} = \sum_{l=0}^{\infty} \frac{\rho^l}{l!} |f|_{l,k+\alpha}, \quad \|\dot{f}\|_{\rho,k+\alpha} = \sum_{l=0}^{\infty} \frac{\rho^l}{l!} |\dot{f}|_{l,k+\alpha}.$$

For analytic f , the series $\|f\|_{\rho,k+\alpha}$ and $\|\dot{f}\|_{\rho,k+\alpha}$ converge for sufficiently small ρ . Indeed, since f and b_{ij} are analytic, we have

$$\|D^\beta f\|_{C_{k+\alpha}(G)} \leq M_1 R_1^{|\beta|} |\beta|!; \quad \|D^\beta b_{ij}\|_{C_{k+\alpha}(G)} \leq M_2 R_2^{|\beta|} |\beta|!.$$

The convergence of $\|f\|_{\rho,k+\alpha}$ will follow from

$$\left\| D^\beta \left(\prod_{\nu=1}^l T_{j_\nu} \right) f \right\|_{C_{R+\alpha}(G)} \ll M R^{|\beta|} \lambda^l (|\beta| + l)! \quad (1)$$

Choose $R > \max(R_1, R_2)$, $\lambda > M_2 R^2 n / (R - R_2)$, $M \geq M_1$. We shall prove the inequality by induction on l . Suppose that it is true for $l \leq \sigma - 1$. Then

$$\begin{aligned} \left\| D^\beta \left(\prod_{\nu=1}^{\sigma} T_{j_\nu} \right) f \right\|_{C_{R+\alpha}(G)} &\ll \sum_{i=1}^n \left\| D^\beta \left(b_{ij_1} D_j \left(\prod_{\nu=2}^{\sigma} T_{j_\nu} \right) f \right) \right\|_{C_{R+\alpha}(G)} \\ &\ll \sum_{i=1}^n \sum_{p+q=\beta} \binom{\beta}{p} \|D^p b_{ij_1}\|_{C_{R+\alpha}(G)} \left\| D^q D_i \left(\prod_{\nu=2}^{\sigma} T_{j_\nu} \right) f \right\|_{C_{R+\alpha}(G)} \\ &\ll M R^{|\beta|} \lambda^\sigma (|\beta| + \sigma)! \frac{M_2 R^2 n}{\lambda} \sum_{m=0}^{|\beta|} \frac{|\beta|! (|\beta| + \sigma - m)!}{(|\beta| - m)! (|\beta| + \sigma)!} \left(\frac{R_2}{R} \right)^m \\ &\ll M R^{|\beta|} \lambda^\sigma (|\beta| + \sigma)! \frac{M_2 R^2 n}{\lambda(R - R_2)} \ll M R^{|\beta|} \lambda^\sigma (|\beta| + \sigma)!. \end{aligned}$$

Thus, if inequality (1) is valid for $l = 1$, then it is valid for all l . The case $l = 1$ is verified separately, and the proof of the inequality is carried out in the same way.

Since an analytic function defined on $\dot{G}$ can be analytically continued inward, the convergence of $\|f\|_{\rho, k+\alpha}$ follows from the preceding.

We note that from the convergence of the series $\|f\|_{\rho, k+\alpha}$ for $\rho \leq \rho_0$, generally speaking, analyticity of f in G does not follow. Consider, for example, the strip $G = \{x : -\infty < x_1, \dots, x_{n-1} < \infty, 0 < x_n < 1\}$. Put $T_j = D_j$, $j = 1, 2, \dots, n-1$. Then from the convergence of the series $\|f\|_{\rho, k+\alpha}$ analyticity of f with respect to $x_1, x_2, \dots, x_{n-1}$ will follow, but not with respect to x_n .

We now formulate the basic properties of the norms in the form of lemmas, the proof of which will be given at the end of the paragraph.

Lemma 1. For functions $f, g \in C_{\infty, k+\alpha}$ the estimates

$$\|T_j f\|_{\rho, k+\alpha} \ll \frac{\partial}{\partial \rho} \|f\|_{\rho, k+\alpha}, \quad (2)$$

$$\|fg\|_{\rho, k+\alpha} \ll \|f\|_{\rho, k+\alpha} \|g\|_{\rho, k+\alpha}. \quad (3)$$

hold. Analogous inequalities, by virtue of assumption 1 on the operators T_j , are also valid for $\|\cdot\|_{\rho, k+\alpha}$.

For convenience, introduce $q_k(\rho)$ and $q(\rho)$ so that

$$\sum_{i=1}^n \|D_m b_{ij}\|_{\rho, k+\alpha} \ll q_k(\rho); \quad 1 \leq m \leq n, \quad 1 \leq j \leq s; \quad (1')$$

$$q_k(\rho) \ll q(\rho), \quad 0 \leq k \leq r-1, \quad (1')$$

Lemma 2. Let $f \in C_{\infty, k+1+\alpha}$. Then

$$\|D_j f\|_{\rho, k+\alpha} \ll \exp \left\{ \int_0^\rho q_k(\xi) d\xi \right\} \|f\|_{\rho, k+1+\alpha}. \quad (4)$$

For the differential operator

$$a(x, D) = \sum_{|\beta| \leq m} a_\beta(x) D^\beta$$

define

$$\|a\|_{\rho, k+\alpha} = \sum_{|\beta| \leq m} \|a_\beta\|_{\rho, k+\alpha}.$$

Denote

$$\|[a]f\|_{\rho, k+\alpha} = \sum_{l=0}^{\infty} \frac{\rho^l}{l!} \max_{j_\nu} \left\| \left[a, \prod_{\nu=1}^l T_{j_\nu} \right] f \right\|_{C_{k+\alpha}(G)},$$

where

$$\left[a, \prod_{\nu=1}^l T_{j_\nu} \right] = a \prod_{\nu=1}^l T_{j_\nu} - \left(\prod_{\nu=1}^l T_{j_\nu} \right) a.$$

Lemma 3. If $f \in C_{\infty, k+m+\alpha}$, $a \in C_{\infty, k+\alpha}$, then for $k+m \leq r$ we have

$$\|[a]f\|_{\rho, k+\alpha} \ll \exp \left\{ m \int_0^\rho q(\xi) d\xi \right\} [\|a\|_{\rho, k+\alpha} - \|a\|_{0, k+\alpha} + m\rho q(\rho)\|a\|_{\rho, k+\alpha}] \|f\|_{\rho, k+m+\alpha}. \quad (5)$$

The proof of Lemmas 1-3 is not given for lack of space.

§ 2. The Dirichlet problem

Theorem 1. Let $G \subset R^n$ be a bounded domain with analytic boundary $\dot{G}$, and let $L(x, D)$ be a uniformly elliptic differential operator of second order with coefficients from $C_{\infty, k-2+\alpha}$. Then for $u \in C_{\infty, k+\alpha}$ the estimate

$$\|u\|_{\rho, k+\alpha} \ll \frac{c}{1 - c[\rho + A(\rho)]} [\|Lu\|_{\rho, k-2+\alpha} + \|\dot{u}\|_{\rho, k+\alpha} + \|u\|_{C_0(G)}], \quad (6)$$

holds, where

$$A(\rho) = \exp \left\{ 2 \int_0^\rho q(\xi) d\xi \right\} [\|L\|_{\rho, k-2+\alpha} - \|L\|_{0, k-2+\alpha} + 2\rho q(\rho)\|L\|_{0, k-2+\alpha}].$$

Here $q(\rho)$ is the formal power series from (1') with $r = 2$. The constant c depends on k, α, n , the boundary $\dot{G}$, the ellipticity constant of the operator L , and the norms of the coefficients of the operator L in $C_{k-2+\alpha}(G)$.

The proof is based on the well-known Schauder estimate (2):

$$|v|_{0, k+\alpha} \ll c [|Lv|_{0, k-2+\alpha} + |\dot{v}|_{0, 0}]. \quad (7)$$

Putting $v = \left(\prod_{\nu=1}^l T_{j_\nu}\right) u$ and using Proposition 1 for the operators T_j , we obtain the inequality

$$|u|_{l,k+\alpha} \ll C \left[\max_{j_\nu} \left| L \left(\prod_{\nu=1}^l T_{j_\nu} \right) u \right|_{0,k-2+\alpha} + |\dot{u}|_{l,k+\alpha} + |u|_{l,0} \right],$$

from which it follows that

$$\|u\|_{\rho,k+\alpha} \ll c \left[\|Lu\|_{\rho,k-2+\alpha} + \|[L]u\|_{\rho,k-2+\alpha} + \|\dot{u}\|_{\rho,k+\alpha} + \|u\|_{\rho,0} \right].$$

Since

$$\|u\|_{\rho,0} \ll |u|_{0,0} + \text{const} \cdot \rho \|u\|_{\rho,1} \ll \|u\|_{C_0(G)} + \text{const} \cdot \rho \|u\|_{\rho,k+\alpha},$$

(5) immediately gives (6).

Theorem 2. Suppose that the hypotheses of Theorem 1 are satisfied. Then

$$\|D_j u\|_{\rho,k-1+\alpha} \ll \frac{c}{1 - c[\rho + A(\rho)]} \exp \left\{ \int_0^\rho q(\xi) d\xi \right\} \left[\|Lu\|_{\rho,k-2+\alpha} + \left(1 + \frac{\partial}{\partial \rho}\right) \|u\|_{\rho,k-1+\alpha} \right], \quad (8)$$

where the constant c depends on the parameters indicated in Theorem 1.

Proof. We shall use Proposition 2 for the operators T_j :

$$\|u\|_{C_{k+\alpha}(\dot{G})} \ll \text{const} \cdot \left[\|u\|_{C_{k-1+\alpha}(\dot{G})} + \sum_{j=1}^s \|T_j u\|_{C_{k-1+\alpha}(\dot{G})} \right]$$

Consequently,

$$\|\dot{u}\|_{\rho,k+\alpha} \ll \text{const} \cdot (1 + \partial/\partial \rho) \|\dot{u}\|_{\rho,k-1+\alpha}. \quad (9)$$

Since

$$\|u\|_{C(G)} \ll \|u\|_{\rho,k-1+\alpha}; \quad \|\dot{u}\|_{\rho,k-1+\alpha} \ll \|u\|_{\rho,k-1+\alpha},$$

then, combining (4), (6), and (9), we obtain (8).

Remark 1. If the uniqueness theorem for the solution of the Dirichlet problem is valid for the operator L , then instead of (7) one may use the estimate

$$|v|_{0,k+\alpha} \ll c [\|Lv\|_{0,k-2+\alpha} + |v|_{0,k+\alpha}]. \quad (10)$$

Then

$$\|u\|_{\rho, k+\alpha} \ll \frac{c}{1 - cA(\rho)} [\|Lu\|_{\rho, k-2+\alpha} + \|\dot{u}\|_{\rho, k+\alpha}]. \quad (11)$$

Combining (9) and (11), we have

$$\|D_j \dot{u}\|_{\rho, k+\alpha} \ll \left(1 + \frac{\partial}{\partial \rho}\right) \left[\frac{c}{1 - cA(\rho)} (\|Lu\|_{\rho, k-2+\alpha} + \|\dot{u}\|_{\rho, k+\alpha}) \right]. \quad (12)$$

Remark 2. Estimates (6) and (8) are valid for unbounded domains if inequality (7) holds. Conditions sufficient for the existence of (7) are given in ⁽²⁾. If the maximum principle is valid in such domains, then inequalities (11) and (12) hold.

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