

CONSTRUCTION OF A SEQUENCE OF STRONGLY INDEPENDENT SUPERINTUITIONISTIC PROPOSITIONAL CALCULI

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1968

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Abstract

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UDC 517.11

MATHEMATICS

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CONSTRUCTION OF A SEQUENCE OF STRONGLY INDEPENDENT SUPERINTUITIONISTIC PROPOSITIONAL CALCULI

(Presented by Academician P. S. Novikov on 12 X 1967)

By a **superintuitionistic logic** we shall mean a set of propositional formulas in the language with four connectives $\&, \supset, \vee, \neg$, containing the intuitionistic axioms and closed under the rules of substitution and *modus ponens*. A logic is a **calculus** if it is obtained by adding to the intuitionistic axioms a finite number of additional ones and closing under the rules of inference.

Logics in a sequence $N_1, \dots, N_k, \dots$ will be called **strongly independent** (see ⁽¹⁾) if, for every i , N_i is not contained in the logic generated by all N_j with $j \neq i$. The models of the intuitionistic propositional calculus studied in ⁽²⁾ under the name of pseudo-Boolean algebras will be called Brouwer algebras*. A Gödelian Brouwer algebra (see ⁽³⁾) will be called (in the finite case) a Brouwer algebra with a unique immediate predecessor of the greatest element**. By a Λ -algebra we shall hereafter mean a finite Gödelian Brouwer algebra α in which there is a system of three generators a_1, a_2, a_3 and the following relations hold:

- 1) $\neg(a_1 \& a_2) = \neg(a_2 \& a_3) = \neg(a_1 \& a_3) = (\neg\neg a_3 \supset a_3) =$
 $= (\neg\neg a_2 \supset a_2) = \neg\neg(a_1 \vee a_2 \vee a_3) = V;$
- 2) $\neg a_3 \vee \neg a_2 \vee (\neg\neg a_1 \supset a_1) = \omega.$

Here V is the greatest element of α , and ω is its unique immediate predecessor.

In ⁽³⁾, for every Gödelian Brouwer algebra α a formula X_α was introduced with the property that an arbitrary formula F is refutable on α (see ⁽³⁾, bearing in mind that in α the greatest element V is singled out) if and only if $F \vdash X_\alpha$ (Theorem 2 of ⁽³⁾; the symbol $\vdash$ denotes derivability of X_α from F using the intuitionistic axioms, *modus ponens*, and substitution). In ⁽³⁾ the following ordering of Gödelian Brouwer algebras is also introduced (isomorphic algebras being identified): $\alpha \leq \beta$ if and only if $X_\alpha \vdash X_\beta$. The following two lemmas ensure the possibility of constructing a sequence of strongly independent calculi.

Lemma 1. *Two Λ -algebras are either incomparable or isomorphic.*

Diagram 1

Figure 1: Diagram 1

Lemma 2. *There exist Λ -algebras with an arbitrarily large number of elements.*

As for the second of these lemmas, the desired Λ -algebras can be constructed as shown in diagram 1 (the distinguished element V is at the top).

It follows from these lemmas that there exists a sequence $\{\alpha_i\}$ of incomparable Λ -algebras. It is convenient for us to assume that $\{\alpha_i\}$ even contains all noniso-

* In ⁽³⁾ they were called implicative structures, which was not quite exact, since it did not reflect the presence of the operation $\neg$. The terminology has not yet become fixed, but the term chosen in ⁽²⁾ seems unsuccessful.

** In ⁽²⁾ an equivalent notion of a strongly compact pseudo-Boolean algebra is introduced.

morphic Λ -algebras (being a Λ -algebra is a decidable property, as is isomorphism of finite Brouwer algebras). Then X_{α_i} and a_i determine the desired calculi N_i .

Indeed, for $j \neq i$ all X_{α} are generally valid on a_i , in view of the incomparability of a_j and a_i , while X_{α} is refuted on a_i , whence X_{α_i} cannot be contained in the logic generated by all X_{α_j} ($j \neq i$).

Diagram 1

Corollary 1 (classical). *There exists a continuum of distinct superintuitionistic logics.*

Corollary 2. *There exists a superintuitionistic logic that is not a calculus. (Such is the logic L generated by all X_{α} .)*

Corollary 3. *There exists a strictly increasing sequence M_i of superintuitionistic calculi (M_i is determined by the axiom $\bigwedge_{i=1}^j X_{\alpha_i}$).*

The logic L also has the following property. The formula

$$\begin{aligned} F \equiv & \neg(a_1 \& a_2) \& \neg(a_2 \& a_3) \& \neg(a_1 \& a_3) \& \\ & \& (\neg\neg a_3 \supset a_3) \& (\neg\neg a_2 \supset a_2) \& \neg\neg(a_1 \vee a_2 \vee a_3) \supset \\ & \supset \neg a_3 \vee \neg a_2 \vee (\neg\neg a_1 \supset a_1) \end{aligned}$$

is not contained in L , but there is no finite Brouwer algebra a on which F would be refuted while all formulas of L would be generally valid.

The first assertion follows from the fact that in a derivation of F we can use only a finite number of formulas X_{α_i} , but, taking a_j distinct from all such a_i , we have: F is refuted on a_j , while all X_{α_i} are generally valid on a_j .

If, however, F is refuted on some finite Brouwer algebra α on which all Λ_{α_i} are generally valid, then, after applying a finite number of times the passage from the algebra to some homomorphic image of it or to some subalgebra of it, we could find a Λ -algebra β with the same properties, but among the a_i there is an algebra isomorphic to it, so that $X_{\alpha_i} \in L$ and X_{α_i} is refuted on β , which contradicts the assumption.

It follows from this theorem that Theorem 3, 4 formulated in ⁽¹⁾ is false. The error in the proof (independently noticed by A. V. Kuznetsov and by me) consists in the implicit application of the relation

$$\overline{\left(L \cap \bigcup_{i=1}^{\infty} L_i\right)} = \bigcap_{i=1}^{\infty} \overline{(L \cup L_i)}.$$

(Here L, L_i are logics, and the bar denotes closure with respect to the rules of inference), which is not always satisfied, as follows from the present work.

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Received
14 IX 1967

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Note: Figure translations are in progress. See original paper for figures.

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