

# SOME REMARKS ON LOGIC OF HIGHER ORDERS

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**Abstract**

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*MATHEMATICS*

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## **SOME REMARKS ON LOGIC OF HIGHER ORDERS**

*(Presented by Academician P. S. Novikov, 13 IV 1967)*

In this note the results of <sup>(1,2)</sup> are strengthened.

We shall say that an ordinal  $\alpha$  is **defined by a formula**  $\sigma$  (of finite or infinite order) if  $\sigma$  is true on all sets well ordered by type  $\alpha$ , and only on them.

By induction we define the notion of a generally definable ordinal:

1. An ordinal definable by a formula of finite order is a generally definable ordinal.
2. If an ordinal  $\alpha$  is defined by a formula such that the types of all variables occurring in it are generally definable ordinals, then  $\alpha$  is a generally definable ordinal.

We shall call a formula **generally definable** if the types of all variables occurring in it are generally definable.

**Theorem 1.** *For every generally definable formula  $\sigma$  one can effectively construct a second-order formula  $\Phi(\sigma)$  such that the truth (satisfiability, categoricity) of  $\sigma$  is equivalent to the truth (satisfiability, categoricity) of  $\Phi(\sigma)$ .*

This theorem strengthens Theorem 5 of <sup>(2)</sup> and, consequently, Hintikka's result <sup>(3)</sup> on the reduction of type theory. From this theorem and a well-known result of A. Tarski <sup>(4)</sup> it follows that

**Theorem 2.** *For every effective enumeration of second-order formulas, the set of numbers of true second-order formulas is not definable in arithmetic by any generally definable formula.*

Let, as in <sup>(1,2)</sup>,  $k(\sigma)$  be the least cardinal in the spectrum of the formula  $\sigma$ , and let  $k_0$  be the upper bound of the set  $\{k(\sigma)\}$ , where  $\{\sigma\}$  is the collection of all satisfiable second-order formulas. It is unknown whether  $k_0$  coincides with the upper bound of the set  $\{k(\tau)\}$ , where  $\{\tau\}$  is the set of all satisfiable formulas whose orders are less than  $k_0$ . In the case of coincidence, one can, assuming the axiom of choice, prove that for every ordinal  $\alpha < k_0$  the set of numbers of true second-order formulas is not definable in arithmetic by any formula of order  $\alpha$ .

At the same time, the set of numbers of true second-order formulas is definable in arithmetic by a formula of order  $(k_0 + 1)$ .

Let us express the result established by Theorem 2 in another form. We shall call a structure  $\mathfrak{S}$  **generally definable** if there exists a generally definable formula true on all structures isomorphic to  $\mathfrak{S}$ , and only on them. We shall call a structure **arithmetized** if in it there is defined a sequence, of type  $\omega$ , of elements  $\Delta_0, \Delta_1, \dots$ . The set of natural numbers  $N$  will be called **definable** (generally definable) in an arithmetized structure  $\mathfrak{S}$  if there exists a formula of finite order (a generally definable formula)  $\sigma$  in the signature  $\mathfrak{S}$  such that  $n \in N$  is equivalent to the truth of  $\sigma(\Delta_n)$  for every natural  $n$ . For every effective enumeration of second-order formulas, the set of numbers of true second-order formulas is not definable in any arithmetized structure definable by a second-order formula

order. If some set of natural numbers is generally definable in a generally definable arithmetized structure, then it is definable by a second-order formula in some structure definable by a second-order formula. Hence it follows that the set of numbers of true second-order formulas will not be generally definable in any generally definable arithmetized structure. Hence, and from the known theorem of Zykov <sup>(5)</sup>, it follows:

**Theorem 3.** *For every effective enumeration of second-order formulas, the set of numbers of true second-order formulas of the form*

$$(\exists X)(Y)\gamma,$$

where  $X$  is a binary and  $Y$  a unary predicate variable, and  $\gamma$  contains no quantifiers over predicate variables, is not generally definable in any generally definable arithmetized structure.

We shall call the **genus of a generally definable formula** the set of types of all variables occurring in it. We shall call the **genus of a system**  $\Sigma$  of generally definable formulas the union of the set of genera of the formulas from  $\Sigma$ .

The following theorem, proved with the aid of Theorem 1, generalizes Theorem 1 from <sup>(2)</sup>.

**Theorem 4.** *Let  $\Sigma$  be a system of generally definable formulas with a finite set of nonlogical constants such that the set of degrees of formulas defining ordinals from the genus  $\Sigma$  is bounded by a generally definable ordinal, and the set of numbers of formulas from  $\Sigma$  (for some effective enumeration of the set of generally definable formulas) is defined in arithmetic by a generally definable formula. Then the system  $\Sigma$  is equivalent to some generally definable formula  $\sigma$ .*

Without giving the exact formulation, in view of its cumbersomeness, let us only note that the relation between the genus  $\Sigma$  and the genus  $\sigma$  is analogous to the

relation between the degrees of a definable system of formulas and an equivalent formula, established by Theorem IV of <sup>(1)</sup>.

We shall say that a set of ordinals  $\mathfrak{D}$  is **defined by a formula**  $\sigma$  if  $\sigma$  is true on all sets from  $\mathfrak{D}$  that are fully ordered by type, and only on them. We shall call a set of ordinals **generally definable** if it is defined by some generally definable formula. With the aid of Theorems 1 and 4 one proves:

**Theorem 5.** *For every generally definable formula  $\sigma$  and every generally definable set  $\mathfrak{D}$  of generally definable ordinals, one can effectively construct second-order formulas  $\Phi(\sigma)$  and  $\Psi(\sigma)$  such that:*

- 1) *the equivalence of  $\sigma$  to some generally definable formula whose genus is included in  $\mathfrak{D}$  is equivalent to the truth of  $\Phi(\sigma)$ ;*
- 2) *the equivalence of  $\sigma$  to some system of generally definable formulas whose genus is included in  $\mathfrak{D}$  is equivalent to the truth of  $\Psi(\sigma)$ .*

This theorem generalizes and strengthens Theorem VI of <sup>(1)</sup>. It implies, in particular, that for every generally definable formula  $\sigma$  and every natural number  $n$  one can effectively construct second-order formulas  $\Phi(\sigma)$  and  $\Psi(\sigma)$  such that the equivalence of  $\sigma$  to some formula of the  $n$ -th order is equivalent to the truth of  $\Phi(\sigma)$ , and the equivalence of  $\sigma$  to some system of formulas of the  $n$ -th order is equivalent to the truth of  $\Psi(\sigma)$ .

The method of proof of Theorem 1 (see also the proof of Theorem 1.1 from <sup>(1)</sup>) shows that the problem of characterizing the spectra of generally definable formulas reduces to the problem of characterizing the spectra of second-order formulas. The complexity of the latter problem is indicated, in part—

...that in the sentence of the strict axiom of infinity there occurs a formula of the 2nd order whose spectrum has the greatest cardinal, greater than  $k_0$ . With the aid of Theorems 1 and 4 one proves

**Theorem 6.** *For every recursively definable set  $\mathfrak{D}$  of recursively definable ordinals there exists a formula of the 2nd order whose spectrum is not predicatively definable in the sense of (2) by any formula whose genus is included in  $\mathfrak{D}$ .*

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*Note: Figure translations are in progress. See original paper for figures.*

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