

# ON MASLOV' S CANONICAL OPERATOR FOR HYPERBOLIC EQUATIONS

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**Abstract**

**Full Text**

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**MATHEMATICS**

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## ON MASLOV' S CANONICAL OPERATOR FOR HYPERBOLIC EQUATIONS

*(Presented by Academician A. N. Tikhonov on 4 IV 1968)*

Maslov's canonical operator  $(^1)^*$ , adapted to the study of solutions of equations of hyperbolic type, is obtained if, as the space  $H$ , one takes the space  $L^2(E^1) \ni \varphi(\tau)$ , and, as the operator  $A$ , the operator  $-i d/d\tau$ . In this case one may say that the space  $W_s$ , containing the range of the canonical operator  $K^\alpha$ , consists of functions differentiable  $s$  times with respect to  $q$  and  $\tau$ , defined up to functions having an indefinitely large number of derivatives with respect to  $q$  and  $\tau$ . For applications to differential equations it is sufficient to know the result of the action of the canonical operator only for  $\tau$  belonging to a neighborhood of zero. We shall introduce such realizations of the canonical operator whose values, for fixed  $q$ , differ by a function of  $\tau$  holomorphic at  $\tau = 0$ . As an instrument we shall use the generalized method of stationary phase.

Let  $f(x)$  be a function holomorphic at the point  $x_0 \in E^n$ ; let  $g(x)$  be a function with values in  $L^2(E^1)$ , holomorphic at the point  $x_0$ . Suppose that  $x_0$  is an isolated stationary point of the real-valued function  $f(x)$ , and that, for any fixed  $x$  from some neighborhood of the point  $x_0$ , the function  $g(x) = g(x; \tau)$  is holomorphic in  $\tau$  on  $(-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ , together with all its derivatives.

Introduce the following notation. Let

$$F : L^2(E^1) \ni \varphi(\tau) \rightarrow F\varphi(\tau) = \tilde{\varphi}(\lambda) \in L^2(E^1)$$

be the Fourier transform. By  $e_\varepsilon$  we shall denote the operator in  $L^2(E^1)$  which reduces to multiplication by the function  $e_\varepsilon(\tau) = 1$  for  $|\tau| < \varepsilon$  and 0 for  $|\tau| > \varepsilon$ , and by  $B_\varepsilon$  the operator in  $L^2(E^1)$  acting according to the formula

$$B_\varepsilon \varphi(\tau) = i e_\varepsilon e^{\delta \tau} \int_0^\infty e^{-\delta y} \varphi(y) dy.$$

For brevity, instead of  $R(A, -i\delta) = (A + i\delta)^{-1}$  we shall write simply  $R$ ;  $\delta$  will always be regarded as positive. The operator  $[-i(A + i\delta)]^{-1/2}$  is by definition equal to

$$F^{-1}[-i(\lambda + i\delta)]^{-1/2} F,$$

where  $[-i(\lambda + i\delta)]^{1/2}$  denotes  $e^{-i\pi/4}(\lambda + i\delta)^{1/2}$  for  $\lambda > 0$  and  $e^{i\pi/4}(|\lambda| - i\delta)^{1/2}$  for  $\lambda < 0$ ;  $(|\lambda| \pm i\delta)^{1/2} \rightarrow (|\lambda|)^{1/2}$  as  $\delta \rightarrow +0$ . Similarly,

$$[i(A - i\delta)]^{-1/2} \stackrel{\text{def}}{=} F^{-1}[i(\lambda - i\delta)]^{-1/2}F,$$

where  $[i(\lambda - i\delta)]^{-1/2}$  denotes  $e^{i\pi/4}(\lambda - i\delta)^{1/2}$  for  $\lambda > 0$  and  $(-\lambda + i\delta)^{1/2}e^{-i\pi/4}$  for  $\lambda < 0$ .

For the case  $n = 1$ , in <sup>(2)</sup> the following is proved.

**Theorem 1.** Let  $f''(x_0) > 0$ ,

$$g_1(\xi) = g(x(\xi))e^{\delta f(x(\xi))} \frac{dx}{d\xi}, \quad g_{n+1}(\xi) = \frac{1}{2} \frac{d}{d\xi} \frac{g_n(\xi) - g_n(x_0)}{\xi - x_0},$$

$$\xi(x) = x_0 + (f(x) - f(x_0))^{1/2}.$$

There exist such positive numbers  $\varepsilon$  and  $\mu$  that the equality is valid

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\* All notation relating to the canonical operator completely coincides with the notation in <sup>(1)</sup>.

$$e_\varepsilon e^{-i(A+i\delta)f(x_0)} \int_{x_0-\mu}^{x_0+\mu} e^{iAf(x)} g(x) dx =$$

$$= \sum_{n=1}^{\infty} \left\{ \sqrt{\pi} \left( \frac{i}{2} \right)^{n-1} (e_\varepsilon R - B_\varepsilon)^{n-1} [-i(A+i\delta)]^{-1/2} g_n(x_0) \right\} + \chi,$$

where  $\chi$  is a function holomorphic on  $(-\varepsilon, \varepsilon)$ , and the series converges uniformly in  $\tau$ . Similarly one proves

**Theorem 2.** Let  $f''(x_0) < 0$ ,

$$g_1(\xi) = g(x(\xi))e^{-\delta f(x(\xi))} \frac{dx}{d\xi}, \quad g_{n+1}(\xi) =$$

$$= -\frac{1}{2} \frac{d}{d\xi} \frac{g'_n(\xi) - g'_n(x_0)}{\xi - x_0}, \quad \xi(x) = x_0 + (f(x_0) - f(x))^{1/2}.$$

There exist positive numbers  $\varepsilon$  and  $\mu$  such that the equality

$$e_\varepsilon e^{i(-A+i\delta)f(x_0)} \int_{x_0-\mu}^{x_0+\mu} e^{iAf(x)} g(x) dx =$$

$$= \sum_{n=1}^{\infty} \left\{ \sqrt{\pi} \left( \frac{i}{2} \right)^{n-1} (e_{\varepsilon} R - B_{\varepsilon})^{n-1} [i(A - i\delta)]^{-1/2} g_n(x_0) \right\} + \chi,$$

where  $\chi$  is a function holomorphic on  $(-\varepsilon, \varepsilon)$ , and the series converges uniformly in  $\tau$ .

For the proof of this theorem the lemma from <sup>(2)</sup> must be modified as follows.

**Lemma.** Let  $f(x)$  and  $g(x)$  be holomorphic on  $[x_0, b]$ ,  $f'(x_0) = 0$ ,  $f''(x_0) < 0$ ,  $f'(x) \neq 0$  for  $x_0 < x \leq b$ ;  $g(x_0)$  and  $g(b)$  are functions from  $L^2(E^1)$ , holomorphic on  $(-\varepsilon - f(x_0) + f(b), \varepsilon - f(x_0) + f(b))$ . Then

$$\begin{aligned} e^{-i(A-i\delta)f(x_0)} \int_{x_0}^b e^{iAf(x)} g(x) dx &= \frac{\sqrt{\pi}}{2} [i(A - i\delta)]^{-1/2} g_1(x_0) + \\ &+ \frac{i}{2} R \int_{x_0}^{b_1} e^{-i(A+i\delta)(\xi-x_0)^2} g_2(\xi) d\xi - \frac{i}{2} R g_1'(0) + \chi, \end{aligned}$$

where  $b_1 = x_0 + (f(x_0) - f(b))^{1/2}$ ,  $\chi$  is a function holomorphic on  $(-\varepsilon, \varepsilon)$ .

We pass to the case  $n > 1$ . Suppose that the matrix

$$R = \left\| \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} \right\|_{i,j=1}^n$$

is nonsingular and the quadratic form corresponding to this matrix is indefinite; the case of a sign-definite form  $R_{ij} z^i z^j$  is reduced directly to Theorems 1 and 2. Let the matrix  $R$  have  $m$  positive and  $n - m$  negative eigenvalues:

$$\lambda_1, \dots, \lambda_m > 0; \quad \lambda_{m+1}, \dots, \lambda_n < 0.$$

In the monograph <sup>(3)</sup>, on pp. 382-385, an analytic mapping  $x \leftrightarrow u, v, \alpha, \beta$ ;  $u \in E^1$ ,  $v \in E^1$ ,  $\alpha \in S^{m-1}$ ,  $\beta \in S^{n-m-1}$  is described such that

$$\int_{\omega} e^{iAf(x)} g(x) dx,$$

where  $\omega$  is some neighborhood of the point  $x_0$ , is equal to

$$\frac{1}{|I|^{1/2}} e^{iAf(x_0)} \int d\Omega_{\alpha} d\Omega_{\beta} \int_0^a \int_0^b e^{iA(u^2-v^2)} \tilde{g}(u, v, \alpha, \beta) du dv.$$

Here  $I = \det R$ ;  $d\Omega_\alpha$  and  $d\Omega_\beta$  are the area elements of the corresponding spheres. Denote by  $G(u, v)$  the function  $\int d\Omega_\alpha d\Omega_\beta \tilde{g}(u, v, \alpha, \beta)$  and introduce recursively the following countable family of functions:

$$G_1(u, v) = G(u, v)e^{\delta u^2}; \quad G_n(u, u) = \frac{i}{2} \frac{\partial}{\partial u} \frac{G_{n-1}(u, v) - G_{n-1}(0, v)}{u};$$

$$G_{n,1}(u, v) = G_n(u, v)e^{\delta v^2}; \quad G_{n,k}(u, v) = -\frac{1}{2} \frac{\partial}{\partial v} \frac{G_{n,k-1}(u, v) - G_{n,k-1}(u, 0)}{v}.$$

**Theorem 3.** The following equality holds

$$\begin{aligned} & e^{-iAf(x_0)} \int_{\omega} e^{iAf(x)} g(x) dx = \\ & = \frac{1}{(|I|)^{1/2}} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \frac{\pi}{4} \left( \frac{i}{2} \right)^{n+k-2} (e_\varepsilon R - B_\varepsilon)^{n+k-2} [i(A - i\delta)]^{-1/2} \times \right. \\ & \quad \left. \times [-i(A + i\delta)]^{-1/2} G_{n,k}(0, 0) \right\} + \chi, \end{aligned}$$

where  $\chi$  is a function holomorphic on  $(-\varepsilon/2, \varepsilon/2)$ .

Without describing all the details here, we give the main course of the proof. Choose  $a \leq (\varepsilon_0/2)^{1/2}$ ,  $\varepsilon < a^2$ . Applying the lemma from (2), we obtain

$$\begin{aligned} \int_0^a e^{iAu^2} G(u, v) du &= \sum_{n=1}^N \frac{\sqrt{\pi}}{2} \left( \frac{i}{2} \right)^{n-1} (e_\varepsilon R - B_\varepsilon)^{n-1} [-i(A + i\delta)]^{-1/2} G_n(0, v) + \\ &+ \left( \frac{i}{2} \right)^N (e_\varepsilon R - B_\varepsilon)^N \int_0^a e^{i(A+i\delta)u^2} G_{N+1}(u, v) du + \chi_N(v). \end{aligned}$$

Integrating with respect to  $v$ , we get

$$\begin{aligned} & \int_0^a \int_0^b e^{iA(u^2-v^2)} G(u, v) dv = \\ & = \sum_{n=1}^N \frac{\sqrt{\pi}}{2} \left( \frac{i}{2} \right)^{n-1} \int_0^b e^{-iAv^2} (e_\varepsilon R - B_\varepsilon)^{n-1} [-i(A + i\delta)]^{-1/2} G_n(0, v) dv + \end{aligned}$$

$$+ \left(\frac{i}{2}\right)^N \int_0^b dv e^{-iAv^2} (e_\varepsilon R - B_\varepsilon)^N \int_0^a e^{i(A+i\delta)u^2} G_{N+1}(u, v) du + \int_0^b e^{-iAv^2} \chi_N(v) dv. \quad (1)$$

It can be shown that the last integral is holomorphic on  $(-\varepsilon, \varepsilon)$ . Applying the lemma to the sum

$$\sum_{n=1}^N \frac{\sqrt{\pi}}{2} \left(\frac{i}{2}\right)^{n-1} \int_0^b e^{-iAv^2} (e_\varepsilon R - B_\varepsilon)^{n-1} [-i(A+i\delta)]^{-1/2} G_n(0, v) dv,$$

we transform it into the form

$$\begin{aligned} & \sum_{n=1}^N \sum_{k=1}^K \left\{ \frac{\pi}{4} \left(\frac{i}{2}\right)^{n+k-2} (e_\varepsilon R - B_\varepsilon)^{n+k-2} [i(A-i\delta)]^{-1/2} [-i(A+i\delta)]^{-1/2} G_{n,k}(0, 0) \right\} + \\ & + \sum_{n=1}^N \left\{ \frac{\sqrt{\pi}}{2} \left(\frac{i}{2}\right)^n (e_\varepsilon R - B_\varepsilon)^{n-1} [-i(A+i\delta)]^{-1/2} \left(\frac{i}{2}\right)^K (e_\varepsilon R - B_\varepsilon)^K \times \\ & \times \int_0^b e^{-i(A-i\delta)v^2} G_{n,K+1}(0, v) dv \right\} + \chi, \end{aligned} \quad (2)$$

where  $\chi$  is holomorphic on  $(-\varepsilon/2, \varepsilon/2)$ . We have used the fact that the commutator of the operators  $e_\varepsilon R - B_\varepsilon$  and  $[i(A-i\delta)]^{-1/2}$  maps any function from  $L^2(E^1)$  into a function holomorphic on  $(-\varepsilon/2, \varepsilon/2)$ . The last sum in (2), for fixed  $N$ , can be made arbitrarily small by choosing  $K$ . The term

$$\left(\frac{i}{2}\right)^N \int_0^b dv e^{-iAv^2} (e_\varepsilon R - B_\varepsilon)^N \int_0^a e^{i(A+i\delta)u^2} G_{N+1}(u, v) du$$

in (1) can be made arbitrarily small by choosing sufficiently large some  $N$ . The proof of the uniform, in  $\tau$  on  $(-\varepsilon, \varepsilon)$ , convergence of the series

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\pi}{4} \left(\frac{i}{2}\right)^{n+k-2} (e_\varepsilon R - B_\varepsilon)^{n+k-2} [i(A+i\delta)]^{-1/2} [-i(A+i\delta)]^{-1/2} G_{n,k}(0, 0)$$

and the estimates of the terms in (1) and (2) are carried out analogously to the way this was done in (2), namely: the estimates of the norms of the functions  $G_n$

and  $G_{n,k}$  are obtained by induction, and then the inequality  $\|(e_\varepsilon R - B_\varepsilon)^n \varphi(\tau)\| \leq e^{n\delta\varepsilon} \sqrt{\varepsilon} \tau^{n-1} [(n-1)!]^{-1} \|\varphi\|$ , valid for every  $\varphi \in L^2(E^1)$ , is used. From the proved theorems on the method of stationary phase there follows the following assertion on the canonical operator for hyperbolic equations. We shall call a special realization of the operator  $V_{i_1 \dots i_k}^{i_m \dots i_k}$  introduced in (1) an operator having the property that, after multiplying it on the left by  $e_\varepsilon$ , one obtains the sum of the series formed from the formal series which defines the operator  $V_{i_1 \dots i_k}^{i_m \dots i_k}$  in (1), by replacing  $R$  by  $e_\varepsilon R - B_\varepsilon$ . We note that the series thus modified is convergent. To a special realization of the operator  $V_{i_1 \dots i_k}^{i_m \dots i_k}$  there corresponds a special realization of the canonical operator.

**Theorem 4.** Let the point  $q_0$  belong to the image of the element  $u_j \in M^n$  under the projection of  $M^n$  onto  $E^n$ ; let all elements of the weight decomposition of unity be holomorphic on  $u_j$ , and let the representatives  $\tilde{\varphi}_{u_j}$  and  $\tilde{\tilde{\varphi}}_{u_j}$  of the function  $\varphi_{u_j}$  be holomorphic and differ from one another by a holomorphic function in a neighborhood of the preimage of the point  $q_0$ , and, moreover, let they themselves and their derivatives for fixed argument be functions of  $\tau$ , holomorphic in some punctured neighborhood of zero, possibly with the exception of zero itself. Then, for two special realizations  $\tilde{K}^{\alpha^0}$  and  $\tilde{\tilde{K}}^{\alpha^0}$  of the canonical operator, the function  $(\tilde{K}^{\alpha^0} \varphi_{u_j})(q_0) - (\tilde{\tilde{K}}^{\alpha^0} \tilde{\tilde{\varphi}}_{u_j})(q_0)$  is holomorphic at the point 0.

**Remark.** The choice of sufficiently small  $\varepsilon$  in the special realizations of the canonical operator is understood.

Thus, one may say that the canonical operator is defined up to functions holomorphic in  $\tau$  in a neighborhood of the point 0.

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## CITED LITERATURE

<sup>1</sup> V. P. Maslov, DAN, **177**, No. 6, 1277 (1967). <sup>2</sup> V. L. Dubnov, UMN, **23**, issue 1 (139), 223 (1968). <sup>3</sup> V. P. Maslov, *Theory of Perturbations and Asymptotic Methods*, Moscow, 1965.

*Note: Figure translations are in progress. See original paper for figures.*

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