

# ON TOPOLOGICAL EQUIVALENTS OF THE BRANCHING HYPOTHESIS\

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## Abstract

## Full Text

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MATHEMATICS

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# ON TOPOLOGICAL EQUIVALENTS OF THE BRANCHING HYPOTHESIS\*

(Presented by Academician P. S. Aleksandrov on 24 IV 1968)

1. The aim of the present paper is to strengthen the results of our preceding paper <sup>(1)</sup>, consisting in the finding of new topological equivalents of the set-theoretic condition  $(\alpha)$  for a strongly inaccessible cardinal number, which we call the branching hypothesis for this number.

Below, as in <sup>(1)</sup>,  $\aleph_\sigma$  is strongly inaccessible and  $\mathfrak{m}$  is an infinite cardinal number. We take the remaining terms and notation from <sup>(1)</sup> without qualification, and add the following new ones. A system of sets  $\Sigma$  is called an **open  $\mathfrak{m}$ -subbase** of a topological space if the collection of intersections of all subsystems in  $\Sigma$  of cardinality  $< \mathfrak{m}$  forms an open base of this space. A partially ordered set  $M$  is called  **$\mathfrak{m}$ -directed** if, for each of its subsets of cardinality  $< \mathfrak{m}$ , there exists an upper bound in  $M$ . In particular,  $\aleph_0$ -directed sets are directed sets in the usual sense of the word. If  $X$  is a topological space, then  $\exp(\mathfrak{m}, X)$  denotes the space on the family of all nonempty closed subsets of  $X$ , whose topology is given by the open base consisting of sets of the form  $B(G_0, \mathfrak{G})$ , where, for an open  $G_0$  and a family  $\mathfrak{G}$  of cardinality  $< \mathfrak{m}$  of open sets in  $X$ ,

$$B(G_0, \mathfrak{G}) = \{F \mid F \subseteq G_0, \forall G \in \mathfrak{G}, F \cap G \supset \Lambda\}.$$

In particular,

$$B(G_0, \{G_1, \dots, G_n\}) = B(G_0, G_1, \dots, G_n)$$

from <sup>(2)</sup>, p. 168, and for  $\mathfrak{m} = \aleph_0$  we have the usual Vietoris exponential space (see, for example, <sup>(2)</sup> or <sup>(3)</sup>).

We formulate the following properties of the number  $\aleph_\sigma$ , supplementing the list from <sup>(1)</sup>:

$\gamma_T(\gamma^+)$ . The Tikhonov  $(\mathcal{T}_\sigma -)$  product of  $\aleph_\sigma$ -bicomact spaces of weight  $\leq \aleph_\sigma$ , taken in number  $\leq \aleph_\sigma$ , is  $\aleph_\sigma$ -bicomact (analogues of Tikhonov's theorem).

$\delta$ . The assertion of Theorem 1 from <sup>(1)</sup>.

$\delta^+$ . If from every cover of a space  $X$  by sets of some open  $\aleph_\sigma$ -subbase of it of cardinality  $\aleph_\sigma$  one can extract a subcover of cardinality  $< \aleph_\sigma$ , then  $X$  is  $\aleph_\sigma$ -bicomact (an analogue of Aleksandrov's lemma).

$\varepsilon$ . If  $X$  is an  $\aleph_\sigma$ -bicomact space of weight  $\aleph_\sigma$ , then  $\exp(\aleph_\sigma, \mathcal{T}_\sigma X)$  is also an  $\aleph_\sigma$ -bicomact space of weight  $\aleph_\sigma$  (an analogue of Vietoris's theorem, <sup>(3)</sup>, p. 161).

$\zeta$ . If  $\{X_\lambda, \pi_\mu^\lambda \mid \text{card}\{\lambda\} = \aleph_\sigma\}$  is an inverse  $\aleph_\sigma$ -directed spectrum of nonempty  $\aleph_\sigma$ -bicomact  $\mathcal{T}_2$ -spaces of weight  $\leq \aleph_\sigma$ , then its limit is nonempty (an analogue of Steenrod's theorem <sup>(4)</sup>, p. 51).

**Theorem.** For strongly inaccessible  $\aleph_\sigma$ , the conditions  $\beta$ ,  $\beta^+$ ,  $\gamma$ ,  $\gamma_T$ ,  $\gamma^+$ ,  $\delta$ ,  $\delta^+$ ,  $\varepsilon$ , and  $\zeta$  are equivalent to the branching hypothesis  $\alpha$ .

Since the equivalence of  $\alpha$ ,  $\beta$ ,  $\beta^+$ ,  $\gamma$ , and  $\delta$  was proved in <sup>(1)</sup>, it is necessary to prove only the remaining implications. Obviously,  $\delta^+ \Rightarrow \delta$ , and the implication  $\alpha \Rightarrow \delta^+$  can be proved without changing anything in the method of proof of  $\alpha \Rightarrow \delta$ .

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\* The results of the paper were reported on 15 II 1968 at a scientific session of the faculty of the Kishinev State University.

(theorem 1 of <sup>(1)</sup>). We shall derive the rest according to the scheme:  $\delta^+ \Rightarrow \varepsilon \Rightarrow \gamma \Rightarrow \alpha \Rightarrow \delta^+ \Rightarrow \nu^\tau \Rightarrow \nu_\tau \Rightarrow \xi \Rightarrow \alpha$ . In the proof of  $\delta^+ \Rightarrow \varepsilon$  we used Frink's idea (see, for example, <sup>(3)</sup>, p. 161), and in the proof of  $\varepsilon \Rightarrow \gamma$  the idea of Maryanovich <sup>(5)</sup> in similar situations.

2. Let  $X$  be a topological space;  $F_0, F_1$  closed in  $X$ ; put

$$\Phi(F_0, F_1) = \{F \mid F \cap F_0 = \Lambda \text{ or } \Lambda \subset F \subset F_1\},$$

where  $F$  is closed in  $X$ . A system of sets shall be called **m**-additive (**m**-multiplicative) if it contains all unions (intersections) of its subsystems of cardinality less than **m**.

(a) If  $F'_0 \cap F''_1 = \Lambda$ , then

$$\Phi(F'_0, F'_1) \cap \Phi(\Lambda, F''_1) = \Phi(\Lambda, F'_1 \cap F''_1).$$

Let  $F$  belong to the left-hand side; then  $F \in \Phi(\Lambda, F''_1)$ , whence  $F \subset F''_1$ ; since  $F'_0 \cap F''_1 = \Lambda$ ,  $F \cap F'_0 = \Lambda$ . But  $F \in \Phi(F'_0, F'_1)$ , whence  $F \subset F'_1$ . Thus  $F \subset F'_1 \cap F''_1$ , and  $F$  belongs to the right-hand side. The reverse inclusion is obvious. It is also obvious that

(b)

$$\bigcap_\lambda \Phi(\Lambda, F_1^\lambda) = \Phi\left(\Lambda, \bigcap_\lambda F_1^\lambda\right).$$

(c) Let **m** be regular and  $X$  an arbitrary **m**-bicomact space. Then any **m**-centered system of families of closed subsets of  $X$  of the form  $\Phi(F_0, F_1)$  has nonempty intersection.

Let  $\tilde{\Phi}$  be our system. Extend  $\tilde{\Phi}$  to  $\tilde{\Phi}^+$  by adding all sets of the form  $\Phi(F_0, F_1)$  which are intersections of subsystems in  $\tilde{\Phi}$  of cardinality  $< \mathfrak{m}$ . Then, by regularity of  $\mathfrak{m}$ ,  $\tilde{\Phi}^+$  contains all  $\Phi(F_0, F_1)$  which are intersections of subsystems in  $\tilde{\Phi}^+$  of cardinality  $< \mathfrak{m}$  (in particular,  $\Phi(\Lambda, X)$ , the intersection of the empty subsystem) and does not contain the empty family  $\Lambda = \Phi(\Lambda, \Lambda)$ . We shall prove that even  $\bigcap \tilde{\Phi}^+ \supset \Lambda$ . Let

$$\tilde{\Phi}^+ = \{\Phi(F_0^\alpha, F_1^\alpha)\}.$$

Let  $\Psi$  be the family of those  $F_1^\alpha$  for which  $F_0^\alpha = \Lambda$ . Then, by (b),  $\Psi$  is  $\mathfrak{m}$ -multiplicative. Put  $F = \bigcap \Psi$  and prove that  $F \in \bigcap \tilde{\Phi}^+$ , i.e.  $\forall \alpha F \in \Phi(F_0^\alpha, F_1^\alpha)$ . Fix  $\alpha$ ; if  $F_0^\alpha \cap F \supset \Lambda$ , then  $F \subset \Phi(F_0^\alpha, F_1^\alpha)$ , so let further  $F_0^\alpha \cap F = \Lambda$ . We first prove that there exists  $F_1^{\alpha'} \in \Psi$  such that  $F_0^\alpha \cap F_1^{\alpha'} = \Lambda$ ; for this suppose the contrary: for all  $F_1^{\alpha'} \in \Psi$ ,  $F_0^\alpha \cap F_1^{\alpha'} \supset \Lambda$ . Then the system

$$\{F_0^\alpha \cap F_1^{\alpha'} \mid F_1^{\alpha'} \in \Psi\}$$

is  $\mathfrak{m}$ -centered, since  $\Psi$  is  $\mathfrak{m}$ -multiplicative. But then

$$\bigcap \{F_0^\alpha \cap F_1^{\alpha'} \mid F_1^{\alpha'} \in \Psi\} = F_0^\alpha \cap F \subset \Lambda$$

by  $\mathfrak{m}$ -bicomcompactness of  $X$ , which contradicts the initial condition. Thus there exists  $F_1^{\alpha'}$  in  $\Psi$  such that

$$F_0^\alpha \cap F_1^{\alpha'} = \Lambda.$$

Then by (a)

$$\Phi(\Lambda, F_1^\alpha \cap F_1^{\alpha'}) = \Phi(\Lambda, F_1^{\alpha'}) \cap \Phi(F_0^\alpha, F_1^\alpha),$$

and the sets on the right belong to  $\tilde{\Phi}^+$ , whence  $\Phi(\Lambda, F_1^\alpha \cap F_1^{\alpha'}) \in \tilde{\Phi}^+$ ,  $F_1^\alpha \cap F_1^{\alpha'} \in \Psi$ ,  $F = \bigcap \Psi \subset F_1^\alpha \cap F_1^{\alpha'} \subset F_1^\alpha$ . Since  $\Psi$  is  $\mathfrak{m}$ -multiplicative and does not contain the empty set,  $\Psi$  is  $\mathfrak{m}$ -centered in the  $\mathfrak{m}$ -bicomcompact space  $X$ , so that  $F_1^\alpha \supset F = \bigcap \Psi \supset \Lambda$  and  $F \in \Phi(F_0^\alpha, F_1^\alpha)$ .

(d) If  $\mathfrak{B}$  is an  $\mathfrak{m}$ -additive open base of the  $\mathfrak{m}$ -bicomcompact space  $X$ , then

$$\{B(H_0, \mathfrak{H}) \mid H_0 \in \mathfrak{B}, \mathfrak{H} \subseteq \mathfrak{B}, \text{card } \mathfrak{H} < \mathfrak{m}\}$$

is an open base in  $\exp(\mathfrak{m}, X)$ .

Let  $B(G_0, \mathfrak{G})$  be a neighborhood of  $F$  in  $\exp(\mathfrak{m}, X)$ . Take a covering of  $F$  by sets from  $\mathfrak{B}$  lying in  $G_0$ , and select from it a subcover of cardinality  $< \mathfrak{m}$ ; the union of the latter, by  $\mathfrak{m}$ -additivity of  $\mathfrak{B}$ , is some  $H_0 \in \mathfrak{B}$ , and  $F \subseteq H_0 \subseteq G_0$ . Further, for each  $G \in \mathfrak{G}$  take  $x(G) \in G \cap F$  and in  $\mathfrak{B}$  choose  $H(G)$ ,  $x(G) \in H(G) \subseteq G$ . Denote the collection of all such  $H(G)$  by  $\mathfrak{H}$ . Then

$$F \in B(H_0, \mathfrak{H}) \subseteq B(G_0, \mathfrak{G}),$$

as was required here.

(e) If  $X$  has weight  $\aleph_\sigma$ , then  $\exp(\aleph_\sigma, X)$  has weight  $\geq \aleph_\sigma$ .

Since  $X$  has weight  $\aleph_\sigma$ , there are at least  $\aleph_\sigma$  closed sets in  $X$ . But from  $F_1 \setminus F_2 \supset \Lambda$  it follows that

$$F_1 \in B(X, CF_2) \setminus B(X, CF_1),$$

so that there are at least  $\aleph_\sigma$  open sets in  $\exp(\aleph_\sigma, X)$ . And a space of weight  $\aleph_\nu < \aleph_\sigma$  has  $\leq 2^{\aleph_\nu} < \aleph_\sigma$  open sets.

From (d) and (e) there follows, independently of the branching hypothesis, the assertion part of  $(\varepsilon)$ :

(f) If  $X$  is an  $\aleph_\sigma$ -bicomcompact space of weight  $\aleph_\sigma$ , then  $\exp(\aleph_\sigma, X)$  has weight  $\aleph_\sigma$ .

Since  $\sum_{\alpha < \sigma} \aleph_\sigma^\alpha = \aleph_\sigma$ , for the proof it is enough to enlarge an open base of  $X$  of cardinality  $\aleph_\sigma$  to an equipotent  $\aleph_\sigma$ -additive one by adding the unions of all its subsystems of cardinality  $< \aleph_\sigma$ .

$$(g) \bigcap_{\lambda} B(G_0^\lambda, \mathfrak{G}^\lambda) = B\left(\bigcap_{\lambda} G_0^\lambda, \bigcup_{\lambda} \mathfrak{G}^\lambda\right); \text{ in particular, } \bigcap_{\lambda} B(G_0, G_1^\lambda) = B(G_0, \{G_1^\lambda\}).$$

3.  $(\delta^+) \Rightarrow (\varepsilon)$ . Let  $X$  be an  $\aleph_\sigma$ -bicomcompact space of weight  $\aleph_\sigma$ ; by  $(\delta)$ ,  $\mathcal{T}^\sigma X$  is also an  $\aleph_\sigma$ -bicomcompact space of weight  $\aleph_\sigma$ , and let  $\mathfrak{B}$  be an  $\aleph_\sigma$ -additive open base of  $\mathcal{T}^\sigma X$  of cardinality  $\aleph_\sigma$  (cf. the proof of (f)). Then, by (d),  $\{B(H_0, \mathfrak{H}) \mid H_0 \in \mathfrak{H}, \mathfrak{H} \subseteq \mathfrak{B}, \text{card } \mathfrak{H} < \aleph_\sigma\}$  is an open base of  $\exp(\aleph_\sigma, \mathcal{T}^\sigma X)$  of cardinality  $\aleph_\sigma$ . Since  $\mathcal{T}^\sigma X$  is a  $\mathcal{T}^\sigma$ -space, by (g) the family  $\{B(H_0, H_1) \mid H_0, H_1 \in \mathfrak{B}\}$  forms an open  $\aleph_\sigma$ -subbase of  $\exp(\aleph_\sigma, \mathcal{T}^\sigma X)$  of cardinality  $\aleph_\sigma$ , and by  $(\delta^+)$ , passing to complements  $CB(H_0, H_1) = \Phi(CH_0, CH_1)$ , and using (c), we obtain the required assertion.

$(\varepsilon) \Rightarrow (\gamma)$ . Replacing, for the disjoint sum  $\cup$ , the sign  $\Pi$ , introduce the topological sum  $\Pi_{\lambda} D_{\lambda} = S$ , where the  $D_{\lambda}$  are two-point discrete spaces,  $\text{card}\{\lambda\} = \aleph_\sigma$ , and consider its one-point extension  $S^+ = S\Pi\infty$ , where neighborhoods of  $\infty$  are taken to be  $O(\infty) = \infty\Pi M$ ,  $M \subseteq S$ ,  $\text{card } CM < \aleph_\sigma$ . It is obvious that a set is closed in  $S^+$  if and only if it either contains  $\infty$ , or has cardinality  $< \aleph_\sigma$ . It is also easy to see that  $S^+$  is an  $\aleph_\sigma$ -bicomcompact  $\mathcal{T}^\sigma$ -space of weight  $\aleph_\sigma$ , and, by  $(\varepsilon)$ ,  $\exp(\aleph_\sigma, S^+)$  has the same properties. Consider in  $\exp(\aleph_\sigma, S^+)$  points of the form  $\Pi_{\lambda} x_{\lambda} \Pi \infty$ , where  $x_{\lambda} = x(\lambda)$  is a choice function for the whole family  $\{D_{\lambda}\}$ ; denote all such points by  $\Xi$  and prove that  $\Xi$  is closed in  $\exp(\aleph_\sigma, S^+)$ . Let  $F \in C\Xi$ ; then for  $F$  there are two possibilities: (1)  $\exists \lambda_0, F \cap D_{\lambda_0} = \Lambda$ ; then  $B(CD_{\lambda_0}, CD_{\lambda_0})$  is a neighborhood of  $F$  not meeting  $\Xi$ ; (2)  $\exists \lambda_0, F \supset D_{\lambda_0}$ ,  $D_{\lambda_0} = \{x_{\lambda_0}, y_{\lambda_0}\}$ ; then  $B(S^+, \{x_{\lambda_0}\}, \{y_{\lambda_0}\})$  is again a neighborhood of  $F$  not meeting  $\Xi$ . Since  $S^+$  is  $\aleph_\sigma$ -bicomcompact,  $\Xi$  is also  $\aleph_\sigma$ -bicomcompact. But the correspondence  $\Pi_{\lambda} x_{\lambda} \Pi \infty \rightarrow \{x_{\lambda}\}$  is a natural homeomorphism of  $\Xi$  onto the  $\mathcal{T}^\sigma$ -product of the family  $\{D_{\lambda}\}$ , whence everything follows.

$(\delta^+) \Rightarrow (\gamma^+) \Rightarrow (\gamma_T)$ . Let  $P = \prod X_\lambda$  be our  $\mathcal{T}^\sigma$ -product; obviously, it is enough to prove the  $\aleph_\sigma$ -bicomcompactness of the product  $P^\sigma = \prod \mathcal{T}^\sigma X_\lambda$  with the stronger topology. By  $(\delta)$ , the  $\mathcal{T}^\sigma X_\lambda$  are also  $\aleph_\sigma$ -bicomcompact and have weight  $\leq \aleph_\sigma$ . Let  $\pi_\lambda$  be the projection of  $P^\sigma$  onto  $\mathcal{T}^\sigma X_\lambda$ , and let  $\mathfrak{B}_\lambda$  be open bases in  $\mathcal{T}^\sigma X_\lambda$  of cardinality  $\leq \aleph_\sigma$ . Since the  $\mathcal{T}^\sigma X_\lambda$  are  $\mathcal{T}^\sigma$ -spaces,  $\{\pi_\lambda^{-1}(B_\lambda) \mid B_\lambda \in \mathfrak{B}_\lambda, \lambda \in \{\lambda\}\}$  is an open  $\aleph_\sigma$ -subbase of  $P^\sigma$  (of cardinality  $\leq \aleph_\sigma$ ). The rest of the argument is carried out analogously to  $(6)$ , p. 143, with the obvious modification. Since the Tikhonov product is a compactification of the corresponding  $\mathcal{T}^\sigma$ -product,  $(\gamma^+) \Rightarrow (\gamma_T)$ .

$(\gamma_T) \Rightarrow (\zeta)$ . Let  $\{X_\lambda, \pi_\mu^\lambda\}$  be our spectrum. The  $\aleph_\sigma$ -centeredness of the system  $\{S_\mu^\lambda \mid \lambda > \mu\}$ ,  $S_\mu^\lambda = \{x \mid x \in \prod X_\lambda, \pi_\mu^\lambda(x_\lambda) = x_\mu\}$ , is proved analogously to the corresponding centeredness in  $(4)$ , p. 51. Since  $\prod X_\lambda$  is  $\aleph_\sigma$ -bicomcompact,  $\bigcap_{\lambda > \mu} S_\mu^\lambda \supset \Lambda$ , as was required.

$(\zeta) \Rightarrow (\alpha)$ . Let  $S$  be a branching system of order  $\omega_\sigma$ . Let  $x \in S$ ; denote by  $p(x)$  the order of  $x$  in  $S$  and put

$$S_\lambda = \{x \mid x \in S, p(x) = \lambda\}, \quad \tau(x) = \sup\{p(y) \mid y \geq x \text{ in } S\}.$$

It is easy to verify the following properties of  $\tau(x)$ : (1) if  $y < x$  in  $S$ , then  $\tau(y) \geq \tau(x)$ , and (2) if  $\tau(y) \geq \lambda$ ,  $y \in S_\mu$ ,  $\mu < \lambda$ , then  $\tau(y) = \sup\{\tau(v) \mid v \in S_\lambda, v \geq y \text{ in } S\}$ . Put  $S_\lambda^0 = \{x \mid x \in S_\lambda, \tau(x) = \omega_\sigma\}$  and prove that all these sets are nonempty. Indeed, otherwise there exists  $\lambda_0 <$

$< \omega_\sigma$  such that for all  $x \in S_{\lambda_0}$ ,  $\tau(x) < \omega_\sigma$ . Since  $\text{card } S_{\lambda_0} < \aleph_\sigma$ , in view of the regularity of  $\omega_\sigma$  we have  $\sup\{\tau(x) \mid x \in S_{\lambda_0}\} < \omega_\sigma$ , and the order of the entire system is  $< \omega_\sigma$ . We now define, for  $\lambda > \mu$ ,  $x \in S_\lambda^0$ ,  $\pi_\mu^\lambda x = y$ , where  $y \in S_\mu$  and  $y \leq x$  in  $S$ . Such a  $y$  exists and is unique by the definition of a branching system. By property (1), for  $\tau(x)$  the mapping  $\pi_\mu^\lambda$  maps  $S_\lambda^0$  into  $S_\mu^0$ , and we shall prove that this mapping is onto.

Suppose the contrary: there exists  $z \in S_\mu^0$ ,  $z \notin \pi_\mu^\lambda(S_\lambda^0)$ . Since  $\tau(z) = \omega_\sigma > \lambda$ , by property (2)

$$\tau(z) = \sup\{\tau(v) \mid v \in S_\lambda \setminus S_\lambda^0, v \geq z \text{ in } S\}.$$

Since all  $\tau(v) < \omega_\sigma$  and  $\text{card } S_\lambda < \omega_\sigma$ , we have  $\tau(z) < \omega_\sigma$ , which is impossible. Considering now all  $S_\lambda^0$  with the discrete topologies, we obtain an inverse spectrum  $\{S_\lambda^0, \pi_\mu^\lambda\}$  satisfying condition  $(\zeta)$ . An element of its nonempty limit gives the sequence in  $S$  required by condition  $(\alpha)$ .

**Remark.** Since assertion (c) of item 2 is proved without using the axiom of choice, Vietoris' theorem on the bicomcompactness of the ordinary exponential space over a bicomcompact space is obtained by using only Alexander's lemma. Since, by Marjanović's method, the Tikhonov product  $\prod_\lambda X_\lambda$  of bicomcompact  $\mathcal{T}_2$ -spaces is embedded as a closed set in the exponential space over the one-point bicomcompactification of P. S. Aleksandrov of the topological sum  $\prod_\lambda X_\lambda$ , it

follows from Alexander's lemma that Tikhonov's theorem already holds for  $\mathcal{T}_2$ -multipliers. This method does not work in the case of  $\mathcal{T}_2$ -multipliers, since then the corresponding set in the exponential space ceases to be closed, which was to be expected, since according to Kelley's theorem (7) the case of  $\mathcal{T}_1$ -multipliers is already exactly equivalent to the axiom of choice (cf. the comments in this connection in (8), p. 274).

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## REFERENCES

1. I. I. Parovichenko, DAN, **174**, No. 1 (1967).
2. K. Kuratowski, *Topology*, 1, 1966.
3. E. Michael, Trans. Am. Math. Soc., **71**, No. 1 (1951).
4. S. Lefschetz, *Algebraic Topology*, 1949.
5. M. Marjanović, Publ. Inst. Math., **6** (20) (1966).
6. J. Kelley, *General Topology*, N. Y., 1955.
7. J. Kelley, Fundam. Math., **37**, 75 (1950).
8. L. Gillman, M. Gerison, *Rings of Continuous Functions*, N. Y., 1960.

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