

AN ANALOG OF THE TOPOLOGY OF A SPACE OF MEASURABLE FUNCTIONS IN A $\backslash(K\backslash)$ -LINEAL WITH UNIT

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Abstract

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MATHEMATICS

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AN ANALOG OF THE TOPOLOGY OF A SPACE OF MEASURABLE FUNCTIONS IN A K -LINEAL WITH UNIT

(Presented by Academician S. L. Sobolev on 17 X 1967)

1. A fundamental system of neighborhoods of zero of a linear topological space X is called a basis. If the space X is separated, is a K -lineal, and has a basis consisting of normal ⁽¹⁾ neighborhoods, then we shall call the basis normal, and X a KT -lineal. By convergence of a net $x_\alpha \rightarrow x$ is meant topological convergence. A KT -lineal has the basic properties of a KN -lineal ⁽¹⁾. A topologically countably complete KT -lineal with unit ^(b) is complete in the sense of ⁽²⁻⁴⁾ (with respect to the norm of bounded elements).
2. Here an analog of the topology of a space of measurable functions is constructed in a certain foundation ⁽¹⁾ Y of an Archimedean K -lineal Z with unit that is complete in the sense of ⁽²⁻⁴⁾ (an a.p.e. K -lineal). Conditions are found under which $Y = Z$. The basis of the KT -lineal is constructed essentially in the same way as was done in ⁽⁵⁾ for the universal semiring, i.e. an extended K -space, but under weaker restrictions on the K -lineal Z and on the topology of its basis I . The topological continuity of the mapping of a KT -lineal X , structurally embedded in Z , is established when the topologies coincide in $X \cap I$.
3. Suppose that in some K -lineal Z there is distinguished a certain system of normal sets W , satisfying the first axiom of separatedness at zero: $\cap W = \{0\}$, and the requirements imposed on a basis of a linear topological space (see, for example, Theorem 1 (1.XI) in ⁽⁶⁾), except for one: some of the sets W may not absorb any elements in Z . Then Z shall be called a generalized KT -lineal with basis $\{W\}$.

If some normal sublineal Y of the generalized KT -lineal Z with unit and basis $\{W\}$ contains its unit and is a KT -lineal with basis $\{W \cap Y\}$, then Y shall be called a KT -foundation in Z .

If each set W absorbs the unit of the generalized KT -lineal Z , then the sublineal

Z_0 of bounded elements is a KT -foundation in Z . It is clear that any KT -foundation $Y \supset Z_0$.

If each set W , moreover, absorbs any element of a certain linear substructure X of the K -linear Z , then there exists a KT -foundation $Y \supset X + Z_0$.

4. An a.p.e. K -linear (see above, item 2) Z is realized ^(3,4) as the K -linear $Z(Q)$ of certain continuous functions, allowed to take infinite values, on a certain bicomact ⁽¹⁾ Q . Within Z there is the K -linear $Z_0 = C(Q)$ of all continuous bounded functions ⁽²⁾. The base I of unit elements e is structurally isomorphic to the system of open-closed sets $E_e \subset Q$.

Suppose there is a system of subsets $\Gamma \subset I$ satisfying the conditions:

$$(\Gamma 1) \quad \text{For any } \Gamma_i \text{ there exists } \Gamma \subset \Gamma_1 \cap \Gamma_2.$$

$$(\Gamma 2) \quad \text{For each } \Gamma_0 \text{ there is } \Gamma \text{ such that if } e_i \in \Gamma, \text{ then } e_1 \vee e_2 \in \Gamma_0.$$

$$(\Gamma 3) \quad \text{All } \Gamma \text{ are normally contained in } I.$$

$$(\Gamma 4) \quad \cap \Gamma = \{0\}.$$

Define in Z the sets W_{Γ_ε} ⁽⁵⁾: $z \in W_{\Gamma_\varepsilon}$ if there exists an element $e \in \Gamma$ such that $\{t; |z(t)| \geq \varepsilon\} \subset E_e$ (here $0 < \varepsilon \leq 1$).

The following assertions hold.

- a) In Z there is a KT -fundament Y with basis $\{Y \cap W_{\Gamma_\varepsilon}\}$, for example $Y = Z_0$.
- b) The system $\{\Gamma\}$ determines a topology of the basis I ,

$$\Gamma = I \cap W_{\Gamma_\varepsilon}.$$

The basis I is a (topological) subspace of the KT -linear Y . Topological convergence of a direction $e_\alpha \rightarrow e$ in I means that the symmetric difference $e_\alpha C e + e C e_\alpha \in \Gamma$ for $\alpha \geq \alpha_0$.

- c) For any finite set of elements y_i of the KT -fundament Y and an arbitrary neighborhood Γ there exists an element $e \in \Gamma$ such that

$$\bigcup_i \{t; y_i(t) = \infty\} \subset E_e.$$

On the set $E_{Ce} = Q \setminus E_e$ the functions $y_i(t)$ are bounded.

- d) In the KT -fundament Y with basis $\{Y \cap W_{\Gamma_\varepsilon}\}$, an abstract function generated ⁽⁴⁾ in an a.p.e. K -lineal Z by a real continuous function of several variables is topologically continuous at all points in whose vicinity the abstract function has meaning in Y .
- e) In order that a certain normal sublineal Y , containing the unit of the given K -lineal Z , be a KT -fundament with basis $\{Y \cap W_{\Gamma_\varepsilon}\}$, it is necessary and sufficient that the following condition be satisfied:
- e') for every element $y \in Y$ and arbitrary neighborhood Γ there is an element $e \in \Gamma$ such that

$$\{t; y(t) = \infty\} \subset E_e.$$

- 5. Let now the bicom pactum Q be completely disconnected ⁽¹⁾. A special case of an a.p.e. K -lineal $Z(Q)$ with a completely disconnected bicom pactum Q is a K_σ -space ⁽¹⁾ with a unit.
- f) For every element $z \in Z$ and arbitrary quantities $\mu, \nu : \mu < \nu \leq +\infty$, there exists an element e such that

$$\{t; z(t) \geq \nu\} \subset E_e, \quad z \geq \mu e.$$

- g) Condition e') is equivalent to the following relation between monotone (r) -convergence of a sequence of unit elements and topological convergence:

$$(\Gamma 5) \quad \text{If } e_n \geq e_{n+1} \xrightarrow{(r)} 0 \text{ in } Y, \text{ then } e_n \rightarrow 0 \text{ in } I.$$

If condition $(\Gamma 5)$ is satisfied for $Y = Z$, then Z is a KT -lineal with basis $\{W_{\Gamma_\varepsilon}\}$.

- h) In order that the whole K -lineal Z be a KT -lineal with basis $\{W_{\Gamma_\varepsilon}\}$, it is sufficient that monotone (o) -convergence of a sequence in I imply topological convergence:

$$(A1) \quad \text{If } e_n \downarrow 0, \text{ then } e_n \rightarrow 0.$$

Condition $(A1)$ is weaker than that indicated in axiom 6 on p. 52 in ⁽⁵⁾.

The topology defined by the basis $\{W_{\Gamma_\varepsilon}\}$ or by its trace $\{Y \cap W_{\Gamma_\varepsilon}\}$ is an analogue of the topology of the space of measurable functions; the sets E_e play the role of measurable sets in the realization of KB -spaces ⁽⁷⁻¹⁰⁾.

- i) If from the convergence $e_\alpha \xrightarrow{(0)} 0$ ($\alpha \in A$) it follows that $e_\alpha \rightarrow 0$, then from the convergence $y_\alpha \xrightarrow{(0)} y$ ($\alpha \in A$) it follows that $y_\alpha \rightarrow y$.
- 6. Let a certain KT -lineal X with normal basis $\{V\}$ be a normal sublineal of some a.p.e. K -lineal Z . Put

$$\Gamma = V \cap I. \quad (1)$$

Then all conditions ($\Gamma 1$ —4) are fulfilled.

If the bicom pactum Q is totally disconnected, then

$$\mu V \subset W_{\Gamma\varepsilon} \quad (0 < \mu < \varepsilon);$$

in Z there is a KT -fundament $Y \supset X + Z_0$; the KT -lineal X with basis $\{V\}$ is topologically continuously embedded in the KT -lineal Y with basis $\{Y \cap W_{\Gamma\varepsilon}\}$.

7. Let some KT -lineal X with unit and normal basis $\{V\}$ be a normal sublineal of some **inner-normal** ⁽⁴⁾ a.p. K -lineal $Z(Q)$ with the same unit (a particular case of such a K -lineal Z is a K_σ -space with unit), and let the neighborhoods Γ be defined by equality (1). Then, in order that the mapping of the KT -lineal X with basis $\{X \cap W_{\Gamma\varepsilon}\}$ into the KT -lineal X with basis $\{V\}$ be topologically continuous, it is necessary and sufficient that the condition be fulfilled (see axiom 8 on p. 52 in ⁽⁵⁾):

($\Gamma 6$) For every neighborhood V_0 there exists a neighborhood Γ such that the algebraic product $\Gamma X \subset V_0$.

If, moreover, the bicom pactum Q is totally disconnected, then the indicated mapping is topologically bicontinuous.

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