

$\backslash(s\backslash)$ -CAPACITY AND THE BEHAVIOR OF THE SOLUTION OF A SECOND-ORDER ELLIPTIC EQUATION WITH DISCONTINUOUS COEFFICIENTS IN A NEIGHBORHOOD OF A BOUNDARY POINT

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Abstract

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MATHEMATICS

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s -CAPACITY AND THE BEHAVIOR OF THE SOLUTION OF A SECOND-ORDER ELLIPTIC EQUATION WITH DISCONTINUOUS COEFFICIENTS IN A NEIGHBORHOOD OF A BOUNDARY POINT

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In this note we consider a linear elliptic equation of the second order

$$Lu \equiv \sum_{i,k=1}^n a_{ik}(x) \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = 0. \quad (1)$$

It is assumed that there exist two positive constants α and M such that, in the domain where the equation is considered,

$$\sum_{i,k=1}^n a_{ik} \xi_i \xi_k \geq \alpha |\xi|^2, \quad \sum_{i=1}^n a_{ii} < M; \quad (2)$$

$$|b_i| < 1, \quad -1 < c \leq 0. \quad (3)$$

By a solution of such an equation we shall mean a classical solution—a continuous function possessing the derivatives entering into the equation and satisfying the equation identically.

In the note the question of the regularity of a boundary point for the Dirichlet problem for this equation will be considered.

If in equation (1) the coefficients of the highest derivatives are smooth or even merely continuous, but with a modulus of continuity satisfying the Dini condition, then the conditions for regularity of a boundary point are the same as for the Laplace equation ⁽¹⁻³⁾. The same will be true if the equation has divergence form (the coefficients in this case may be arbitrary bounded ones) ⁽⁴⁾. In the general case, however, as is shown by the example given in Sec. 3°, this is no longer so.

Definition 1. Let positive numbers α and M be given. A point x^0 of the boundary Γ of a domain D is called α, M -regular with respect to the Dirichlet problem if there exists a continuous function $\omega(r)$, $0 \leq r < r_0$, $\omega(0) = 0$, $\omega(r) > 0$ for $r > 0$, such that, whatever the neighborhood σ of the point x^0 , the domain $D' \subset D$ with boundary Γ' , and the equation (1), for which inequalities (2) hold with the given constants α, M , and whatever the solution $u(x) < 1$ of this equation, continuous on $D' \cup \Gamma'$ and nonpositive on the intersection $\Gamma' \cap \sigma$, there is a neighborhood σ' of the point x^0 , depending only on σ, α , and M , such that $u(x) < \omega(|x - x^0|)$ for $x \in \sigma' \cap D$.

We shall give sufficient conditions for regularity of a boundary point.

1°. **s -capacity.** Let s be a positive number. Let E be a B -set of n -dimensional Euclidean space. Consider all possible measures μ , defined on subsets of the set E , such that

$$\int_E \frac{d\mu(y)}{|x - y|^s} \leq 1 \quad \text{for } x \notin E; \quad (4)$$

the number $C_s(E) = \sup \mu(E)$, where the least upper bound is taken over all measures μ satisfying inequality (4), will be called the s -capacity of the set E .

Lemma 1. The s -capacity of a ball of radius R is not less than R^s .

Lemma 2. Let $s > 1$ and let $\Pi_{\rho, h}$ be the cylinder:

$$\sum_{i=1}^{n-1} x_i^2 < \rho^2, \quad 0 < x_n < h,$$

where $\rho < h$. Then $C_s(\Pi_{\rho, h}) \geq Ch\rho^{s-1}$, where C is a positive constant depending on s .

Lemma 3. Let $s < n$; let E be located inside the ball of unit radius. Then $C_s(E) \geq \eta \text{mes } E$, where $\eta > 0$ is a constant depending on s and n (by $\text{mes } E$ is denoted the Lebesgue measure of the set E).

Lemma 4. For any $a > 0$ and $M > 0$ there exist $s > 0$ and $r_0 > 0$ such that, for every operator L of the form (1) satisfying inequalities (2) and (3), the inequality

$$L(1/|x - x^0|^s) \geq 0$$

holds, where x^0 is a fixed point of R_n , x is a point of the domain where the operator L is defined, and $|x - x^0| < r_0$.

In this case we shall say that the s -capacity is **upper** for the operator L in radius r_0 .

Remark 1. The number s can be made to depend only on the ratio M/a .

The proof of these lemmas is elementary.

2°. Sufficient conditions for regularity of a boundary point. In what follows, by $Q_R^{x^0}$ we shall denote the ball of radius R with center at the point x^0 .

Lemma 5 (main). Let a domain D be situated in the ball $Q_{4R}^{x^0}$, have limit points on the surface of the ball $Q_R^{x^0}$, and intersect the ball $Q_R^{x^0}$. Let H be the intersection of the complement of the domain D with the ball $Q_R^{x^0}$. Let Γ be that part of the boundary of the domain D which is situated strictly inside $Q_{4R}^{x^0}$. Let equation (1) be defined in D . Let $u(x)$ be its solution, positive in D , continuous in $\overline{D}$, and vanishing on Γ . Let the numbers $s > 0$ and $r_0 > 0$ be such that the s -capacity is upper for the operator L in radius r_0 and $8R < r_0$.

Then

$$\max_{\overline{D}} u \geq \left[1 + \frac{\xi}{R^s} C_s(H) \right] \max_{D \cap Q_R^{x^0}} u,$$

where $\xi > 0$ is a constant depending on s .

Proof. Choose an arbitrary $\varepsilon > 0$, and let the measure μ be such that

$$U(x) = \int_H \frac{d\mu(y)}{|x-y|^s} \leq 1 \quad \text{for } x \notin H, \quad \mu(H) > C_s(H) - \varepsilon.$$

Let

$$M = \max_{\overline{D}} u.$$

Put

$$v(x) = M \left[1 - U + \frac{C_s(H)}{(3R)^s} \right].$$

We have $Lv \leq 0$ in D , and v is not less than u on the boundary of the domain D . Therefore

$$\max_{D \cap Q_R^{x^0}} u \leq \max_{D \cap Q_R^{x^0}} v \leq M \left[1 - \frac{C_s(H)}{R^s} \left(\frac{1}{2^s} - \frac{1}{3^s} \right) + \frac{\varepsilon}{(3R)^s} \right],$$

which, by the arbitrariness of ε , proves the lemma.

With the help of Lemma 5 one proves

Lemma 6. Let, in the spherical layer

$$R < |x - x^0| < 4R,$$

there be situated a domain D , having limit points on both spheres $|x - x^0| = R$ and $|x - x^0| = 4R$. Let Γ be that part of the boundary of the domain D which is situated strictly inside the layer. Let equation (1) be defined in D , and let $u(x)$ be its solution, positive in D , continuous in $\overline{D}$, and vanishing on Γ . Let the numbers $s > 0$ and $r_0 > 0$ be such that the s -capacity is upper for the operator L in radius r_0 and $8R < r_0$.

Then

$$\max_{x \in \overline{D}} u(x) > \left[1 + \frac{\zeta C_s(H)}{R^s} \right] \max_{x \in D, |x-x^0|=2R} u(x),$$

where ζ is a constant depending on s and r_0 , and $H = (Q_{3R}^{x^0} \setminus Q_{2R}^{x^0}) \setminus D$.

With the aid of Lemma 5 one proves

Theorem 1. Let there be a domain D with boundary Γ , and let $x^0 \in \Gamma$. Denote by H the complement of the domain D . Let $\alpha > 0$ and $M > 0$ be given numbers, and let $s > 0$ be such that s -capacity is an upper one for every operator L satisfying inequality (2).

Set $\gamma_m = C_s(H \cap Q_{2^{-m}}^{x^0})$, and suppose that

$$\sum_{m=1}^{\infty} 2^{ms} \gamma_m = \infty. \quad (5)$$

Then the point x^0 is α, M -regular, and, if we denote

$$S(t) = \sum_{m=1}^{[t]} 2^{ms} \gamma_m,$$

then for the function $\omega(r)$, for sufficiently small r , the estimate

$$\omega(r) < e^{-aS(|\ln r|)}$$

will hold, where the constant $a > 0$ depends on α and M .

In the case where the coefficients of the equation satisfy the Hölder condition, an analogous result was obtained in [5].

With the aid of Lemma 6 one proves

Theorem 2. Let D, Γ , and x^0 have the same meaning as in Theorem 1. Let equation (1) be defined in D , and let $s > 0$ be such that s -capacity is upper for the operator L . Let γ_m have the same meaning as above, and let condition (5) be fulfilled. Then a bounded solution of equation (1), continuous on Γ in a neighborhood of x^0 , is also continuous at x^0 .

Such a point will be called O -regular.

From Lemma 1 and Theorems 1 and 2 it follows that

Theorem 3. If a boundary point of a domain can be touched by a vertex cone from outside the domain, then such a point is regular (in both senses) for any equation (1), and as $\omega(r)$ one may take r^β for sufficiently small $\beta > 0$ (β depends on α and M , more precisely on their ratio).

Lemma 2 makes it possible to give a milder sufficient condition for regularity:

Theorem 4. Let the point x^0 of the boundary Γ of the domain D possess the following property. One can introduce an orthogonal coordinate system $y_1, \dots, y_n$ with origin at this point such that the points belonging to the cusp

$$\left(\sum_{i=1}^{n-1} y_i^2 \right)^{1/2} < \frac{y_n}{(\ln 1/y_n)^{1/(s-1)}}, \quad 0 < y_n < h \quad (6)$$

for some h , $0 < h < 1$, do not belong to the domain D . Then the point x^0 is O -regular for every equation (1) for which s -capacity is upper. For any α and M to which this s corresponds, the point x^0 is α, M -regular.

This condition corresponds to the Uryson condition [6] for the Laplace equation when $n \geq 4$. It is all the more interesting that when $n = 3$, for a general equation (1), in contrast to the Laplace equation, this condition cannot be substantially weakened, as is shown by the example given in the following section.

If $M/\alpha < n + 2$, then one can find $s < n$ for which s -capacity is upper for the corresponding equations. Lemma 3 permits, in this case, replacing in Theorems 1 and 2 the capacity of the intersection $H \cap Q_{2^{-m}}^{x^0}$ by its Lebesgue measure.

3°. One example of an irregular point. For any $\delta > 0$ one can find a homogeneous elliptic equation with three independent

variables

$$\sum_{i,k=1}^3 a_{ik}(x_1, x_2, x_3) \frac{\partial^2 u}{\partial x_i \partial x_k} = 0,$$

whose coefficients differ from the coefficients of Laplace's equation by less than δ , defined in a domain D , the boundary Γ of which has a point x^0 of the same type as in Theorem 4, but with another exponent of the logarithm in inequality (6):

$$\left(\sum_{i=1}^{n-1} y_i^2 \right)^{1/2} < \frac{y_n}{(\ln 1/y_n)^a},$$

where $a > 0$ is a number depending on δ , and find a solution $u(x)$ of this equation in the domain D such that: $u(x)$ is bounded, $u(x)$ is continuous at all boundary points except x^0 , in a neighborhood of x^0 it assumes on the boundary the value equal to one, and, at the same time,

$$\lim_{x \rightarrow x^0} u(x) < 1.$$

Let $\varepsilon > 0$ be arbitrary and $a > (1 + \varepsilon)/\varepsilon \ln 2$. Put $s = 1 + \varepsilon$. Put

$$u(x_1, x_2, x_3) = \sum_{m=N}^{\infty} \frac{2^{-m(s-1)}}{m^s} \int_{2^{-(m+1)}}^{2^{-m}} \frac{dt}{[x_1^2 + x_2^2 + (x_3 - t)^2]^{s/2}}.$$

The function u is defined outside the positive half-axis x_3 . Choose the number N so that $u(0, 0, 0) < 1/2$, which is possible, since at this point the series converges. The set of points where $u(x_1, x_2, x_3) > 1$ forms a funnel B around the positive half-axis x_3 , containing the funnel

$$\left(\sum_{i=1}^2 x_i^2 \right)^{1/2} < \frac{x_3}{(\ln 1/x_3)^a}, \quad 0 < x_3 < 2^{-N}.$$

Take as the domain D the ball of radius 2^{-N} with center at the origin, from which the funnel $\bar{B}$ has been removed.

We now construct an equation whose solution is $u(x)$. Through every point $x \in D$ draw a plane π through this point and the axis x_3 (at points of the axis x_3 , $x_3 < 0$, one may take as π any plane passing through the axis x_3). Introduce a local orthogonal coordinate system ξ_1, ξ_2, ξ_3 so that the axes ξ_1, ξ_2 lie in the plane π . At this point x , in the coordinates ξ , define the equation as follows:

$$\frac{2}{2 + \varepsilon} \left(\frac{\partial^2 u}{\partial \xi_1^2} + \frac{\partial^2 u}{\partial \xi_2^2} \right) + \frac{\partial^2 u}{\partial \xi_3^2} = 0.$$

Remark 2. The condition of nonnegativity of the coefficients $C(x)$ can be replaced by the condition that this coefficient be bounded from above.

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Note: Figure translations are in progress. See original paper for figures.

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