

NORMAL SEQUENCES AND FINITE AUTOMATA

V. N. Agafonov

1968

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196801.34465>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 519.95:519.211:511

MATHEMATICS

V. N. AGAFONOV

NORMAL SEQUENCES AND FINITE AUTOMATA

(Presented by Academician P. S. Novikov on 15 V 1967)

In a number of works ⁽¹⁻⁶⁾ infinite binary sequences that are "random" in one sense or another are studied. The notion of a random sequence considered in ⁽¹⁻³⁾ is Wald's ⁽²⁾ refined notion of a collective, introduced by von Mises ⁽¹⁾. In ^(5,6) normal sequences are studied. In the present note the connection between these notions is clarified in terms of finite automata.

1. Collective. We shall call a *strategy* a predicate defined on the set of all finite binary sequences (including the empty sequence Λ). A strategy f defines the following procedure for selecting from an infinite binary sequence $x_1x_2x_3 \dots$ a subsequence (possibly finite) $y_1y_2y_3 \dots$. If $z_1z_2z_3 \dots$ is the sequence of values of the strategy f corresponding to the sequence $x_1x_2x_3 \dots$: $z_1 = f(\Lambda)$, $z_2 = f(x_1)$, $z_3 = f(x_1x_2), \dots$, and $i_1 < i_2 < i_3 < \dots$ are the numbers of those places for which $z_{i_k} = 1$, then $y_1 = x_{i_1}$, $y_2 = x_{i_2}$, $y_3 = x_{i_3}, \dots$. The sequence $x_1x_2x_3 \dots$ is called a collective with respect to a set of strategies F , containing the strategy identically equal to one, if for all infinite subsequences selected from $x_1x_2x_3 \dots$ by strategies of the set F , there exists one and the same limit of the relative frequency of ones

$$p = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n x_i \quad (0 < p < 1).$$

2. Normal sequence. An infinite binary sequence $x_1x_2x_3 \dots$ is called normal if there exists a number p ($0 < p < 1$) such that for every positive integer n and every word $a_1a_2 \dots a_n$ of length n in the alphabet $\{0,1\}$, the following condition is satisfied: in the sequence of words $x_1x_2 \dots x_n, x_{n+1}x_{n+2} \dots x_{2n}, x_{2n+1}x_{2n+2} \dots x_{3n}, \dots$ the relative frequency of occurrence of the word $a_1a_2 \dots a_n$ has a limit equal to $p^k(1-p)^{n-k}$, where $k = \sum_{i=1}^n a_i$ is the number of ones in the word $a_1a_2 \dots a_n$.

3. Normal sequence as a collective. In ⁽⁷⁾ it is established that a sequence is normal if and only if it is a collective with respect to the set of strategies possessing the following property: the strategy selects or does not select the

next symbol of the sequence depending on what the preceding l symbols are, where for the given strategy the number l is fixed.

It is clear that for every strategy f of this kind there exists a finite automaton $\langle \{0, 1\}, Q, Q^*, \varphi, q \rangle$ with input alphabet $\{0, 1\}$, set of states Q , subset of selected states $Q^* \subseteq Q$, transition function $\varphi : Q \times \{0, 1\} \rightarrow Q$, and initial state $q \in Q$, which computes the strategy f in the following way:

$$f(x_1 x_2 \dots x_n) = 1 \iff \varphi(\dots \varphi(\varphi(q, x_1), x_2) \dots), x_n) \in Q^*.$$

Hence it follows that a collective with respect to the set of strategies computable by finite automata,

is a normal sequence. The natural question arises whether the converse assertion is true. An affirmative answer to it is given by the following

Theorem. *A normal sequence is a collective with respect to the set of strategies computable by finite automata.*

The proof of the theorem uses the ergodic property of a finite automaton to whose input a sequence of independent random variables is fed, and the following “nonincrease of measure” property possessed by the transformation defined by a strategy on words. Let the p -weight of a word $a_1 \dots a_n$ be the number $p^l(1-p)^{n-l}$, where l is the number of ones in the word $a_1 \dots a_n$ and $0 < p < 1$, and let the p -weight of a subset of words of fixed length be the sum of the p -weights of all words in this subset. Then for any strategy f and any word $a_1 \dots a_k$, the set of words of length $n \geq k$ from which f selects words beginning with the word $a_1 \dots a_k$ has p -weight not exceeding the p -weight of the word $a_1 \dots a_k$.

In conclusion I express my gratitude to Prof. B. A. Trakhtenbrot, who proposed the problem and made a number of valuable comments.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

Received
14 V 1967

REFERENCES

1. P. Mises, *Probability and Statistics*, Moscow, Gosizdat, 1930.
2. A. Wald, *Coll. consacré à la théorie des probabilités*, 1938, p. 79.
3. A. Church, *Bull. Am. Math. Soc.*, **46**, 130 (1940).
4. D. W. Loveland, *Trans. Am. Math. Soc.*, **125**, 497 (1966).
5. A. Copeland, *Am. J. Math.*, **50**, 535 (1928).

6. A. G. Postnikov, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **57** (1960).

7. L. P. Postnikova, *Teoriya veroyatnostei i ee primeneniya*, **6**, 232 (1961).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.