

ON RANDOM PROCESSES GENERATED BY NUMBER-THEORETIC ENDOMORPHISMS

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Abstract

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MATHEMATICS

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ON RANDOM PROCESSES GENERATED BY NUMBER-THEORETIC ENDOMORPHISMS

(Presented by Academician Yu. V. Linnik on 19 II 1968)

1. A. Rényi ⁽¹⁾ studied transformations of the interval $(0, 1)$ defined by the formula $Tx = \{\varphi(x)\}$, where $\{a\}$ denotes the fractional part of the number a , and $\varphi = f^{-1}$, with the function f satisfying one of the conditions A, B and condition C. These conditions are as follows:

A. f is defined and decreases on $[1, +\infty)$, $f(1) = 1$, f is strictly positive, continuous and strictly decreasing on $[1, T)$, and $f = 0$ on $[T, +\infty)$, where T is either a natural number or $+\infty$ (in this case it is meant that $\lim_{t \rightarrow \infty} f(t) = 0$). In addition, $|f(t_2) - f(t_1)| \leq |t_2 - t_1|$, and there exists λ , $0 < \lambda < 1$, such that $|f(t_2) - f(t_1)| \leq \lambda |t_2 - t_1|$ if $1 + f(2) < t_1 < t_2$.

B. f is defined and increasing on $[0, +\infty)$, $f(0) = 0$, f is continuous and strictly increasing on $[0, T]$, and $f = 1$ on $[T, +\infty)$, where T is either a natural number or $+\infty$ (in this case $\lim_{t \rightarrow \infty} f(t) = 1$). In addition, $f(t_2) - f(t_1) \leq t_2 - t_1$, if $0 \leq t_2 < t_1$.

C.

$$\operatorname{ess\,sup}_{0 < x < 1} f'_{E_n}(x) \leq C \operatorname{ess\,inf}_{0 < x < 1} f'_{E_n}(x).$$

Here $E_n = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, where ε_i are numbers from the range of values of the function $[\varphi(x)]$, $0 < x < 1$ ($[a]$ is the integer part of a), $f_{E_n}(x) = f(\varepsilon_1 + f(\varepsilon_2 + \dots + f(\varepsilon_n + x) \dots))$, and the constant C depends neither on E_n nor on n .

It follows from Rényi's results that if f satisfies one of the conditions A, B and condition C, then the sequence of functions $a(T^n x)$, $n \geq 0$, $a(x) = [\varphi(x)]$, generates on $(0, 1)$ the entire Borel σ -algebra, and for the transformation T on $(0, 1)$ there is an invariant measure $\mathbf{P}$, having density $p(x)$ with respect to Lebesgue measure, with

$$1/C \leq p(x) \leq C$$

almost everywhere on $(0, 1)$.

V. A. Rokhlin ⁽²⁾ proved that under these conditions the transformation T of the space $X = (0, 1)$ with measure $\mathbf{P}$ is an exact endomorphism, i.e. that

$$\bigcap_{k=0}^{\infty} M_k = N.$$

Here $M_0 = M$ is the σ -algebra of Lebesgue subsets of $(0, 1)$;

$$M_k = T^{-k}(M)$$

is the σ -algebra of sets differing only by a set of measure 0 from the complete inverse image of some Lebesgue set with respect to T^k ; N is the σ -algebra containing only sets of measure 0 or 1. Hence follows the ergodicity of T , proved in ⁽¹⁾.

The equality

$$\int_A Vg(x) \mathbf{P}(dx) = \int_{T^{-1}(A)} g(x) \mathbf{P}(dx), \quad A \in M,$$

defines a linear operator V , acting in each of the spaces $L_p(\mathbf{P})$ ($1 \leq p \leq \infty$) with norm 1. The operator V , which can be defined for any measure-preserving transformation, has a number of interesting properties. In particular, if T is an exact endomorphism of a space—

space X with measure $\mathbf{P}$, then for $g \in L_p(\mathbf{P})$, $1 \leq p < \infty$,

$$V^n g \xrightarrow{L_p(\mathbf{P})} \int_X g(x) \mathbf{P}(dx).$$

This relation, in the situation considered by us, can be sharpened.

Let us formulate conditions D and E:

D. All the functions f'_{E_n} have bounded variation and

$$\sum_{E_n} \text{var}(f'_{E_n}) \leq K,$$

where K does not depend on n .

E.

$$\text{var}(f'_{E_n}) \leq K \int_0^1 |f'_{E_n}(x)| dx,$$

where K does not depend on E_n or n .

Condition E is stronger than condition D, since

$$\sum_{E_n} \int_0^1 |f'_{E_n}(x)| dx = 1.$$

In Theorems 1, 2, 3 (see below) it is assumed that f satisfies one of conditions A, B and condition C. By $\alpha(n)$ is denoted a function of the form $Ae^{-\lambda n}$, where the constants $A > 0$, $\lambda > 0$ depend only on f .

Theorem 1. *Let g be a function of bounded variation. If condition D is satisfied, then*

$$\operatorname{ess\,sup}_{0 < x < 1} \left| V^n g(x) - \int_0^1 g(u) \mathbf{P}(du) \right| \leq \alpha(n) \operatorname{var}(g).$$

Let us now put $X_k(x) = [\varphi(T^k x)]$ and denote by M_a^b the σ -algebra generated by all the functions X_k , $0 \leq a \leq k \leq b \leq \infty$.

Theorem 2. *Let condition D be satisfied. Then*

$$|\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B)| \leq \alpha(n)\mathbf{P}(B),$$

if $A \in M_0^k$, $B \in M_{k+n}^\infty$, $n \geq 1$.

Theorem 3. *Let condition E be satisfied. Then*

$$|\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B)| \leq \alpha(n)\mathbf{P}(A)\mathbf{P}(B),$$

if $A \in M_0^k$, $B \in M_{k+n}^\infty$, $n \geq 1$.

Remark 1. Condition E is satisfied if, for example, f'' exists and for some $p > 1$

$$\sup_l \int_0^1 \left| \frac{f''(x+l)}{f'(x+l)} \right|^p dx < +\infty.$$

The numbers l run through the range of values of the function $[\varphi(x)]$, $x \in (0, 1)$.

Remark 2. If $\theta > 1$, θ nonintegral, then $f(x) = x/\theta$ does not satisfy condition B. However, as shown in ⁽¹⁾ and ⁽³⁾, the sequence X_k generates the Borel σ -algebra, and the transformation T has an invariant measure. For the case $\theta > 2$ one can prove analogues of Theorems 1 and 2 with $\alpha(n) = A(2/\theta)^n$.

Remark 3. The results of Theorems 1, 3 for $f(x) = 1/x$ give estimates known in the metric theory of continued fractions ⁽⁸⁾. Under more stringent restrictions on f , an estimate of order $Ae^{-\lambda\sqrt{n}}$ was obtained in ⁽⁴⁾.

2. Here probabilistic notation will be used. All mathematical expectations are taken with respect to the invariant measure $\mathbf{P}$. Let $g \in L_2(\mathbf{P})$, $\mathbf{E}g = 0$. The sequence $g(T^k x)$, $k \geq 0$, forms a stationary random process. A number of results are known which make it possible to apply limit theorems to similar processes probability theory, provided only that these processes are asymptotically independent in a certain sense, as, for example, the process X_k in Theorem 2 (for the central limit theorem see, for example, (5), Ch. XVIII). However, one of the conditions for the applicability of such theorems is the relation

$$B_n(g) = \mathbf{E} \left(\sum_{k=0}^{n-1} g(T^k x) \right)^2 \xrightarrow{n \rightarrow \infty} \infty,$$

whose verification is not always simple.

Theorem 4. *If f satisfies the conditions of Theorem 1 and f' is continuous, $g(x) = G([\varphi(x)])$, $\mathbf{E}g = 0$, $\mathbf{E}g^2 < \infty$, then*

$$B_n(g) \xrightarrow{n \rightarrow \infty} \infty.$$

Theorem 5. *If f satisfies the conditions of Theorem 1, g is an unbounded function, Vg has bounded variation, $\mathbf{E}g = 0$, $\mathbf{E}g^2 < \infty$, then*

$$B_n(g) \xrightarrow{n \rightarrow \infty} \infty.$$

Let $q_n(x)$ be the denominator of the n -th convergent of the expansion of the number x into an ordinary continued fraction. With the aid of Theorem 5, for $f(x) = 1/x$,

$$g(x) = \ln \frac{1}{x} - \int_0^1 \ln \frac{1}{x} p(x) dx, \quad \text{where } p(x) = \frac{1}{(1+x) \ln 2},$$

one can prove that

$$\int_0^1 \left(\ln q_n(x) - \int_0^1 \ln q_n(u) p(u) du \right)^2 p(x) dx \xrightarrow{n \rightarrow \infty} \infty.$$

Relying on this relation, one can prove the central limit theorem and the law of the iterated logarithm for $\ln q_n$ (see (7)).

Remark 4. The result of Theorem 4 for $f(x) = 1/x$ and the above relation for $\ln q_n$ were stated without proof by W. Doeblin (6).

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