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Abstract

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MATHEMATICS

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ON INTEGRAL-DIFFERENCE WIENER-HOPF EQUATIONS

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1. In the present note we study integral-difference equations of the form

$$\sum_{j=-\infty}^{\infty} a_j \varphi(t - \delta_j) + \int_0^{\infty} k(t-s) \varphi(s) ds = f(t) \quad (0 \leq t < \infty), \quad (1)$$

where a_j ($j = 0, \pm 1, \dots$) are prescribed complex numbers; δ_j ($j = 0, \pm 1, \dots$) are prescribed real numbers, $k(t)$ ($-\infty < t < \infty$) and $f(t)$ ($0 \leq t < \infty$) are prescribed functions, and $\varphi(t)$ ($0 \leq t < \infty$) is the unknown function. In addition, in (1) it is assumed that, for $\delta_j > 0$, the function $\varphi(t - \delta_j)$ is set equal to zero on the interval $0 \leq t \leq \delta_j$.

Equation (1) is a generalization both of the Wiener-Hopf integral equation and of its discrete analogue. If one puts $a_j = 0$ ($j \neq 0$) and $\delta_0 = 0$, then equation (1) coincides with the Wiener-Hopf integral equation; if $k(t) \equiv 0$ and $\delta_j = j$ ($j = 0, \pm 1, \dots$), then, with respect to the vector-function $\varphi(t) = \{\varphi_1(t), \varphi_2(t), \dots\}$, where $\varphi_j(t) = \varphi(t + j - 1)$ ($0 \leq t \leq 1$), equation (1) is the discrete Wiener-Hopf equation with matrix $\|a_{j-k}\|_{i,k=1}^{\infty}$.

Below equation (1) will be considered in the space $L_p(0, \infty)$ ($1 \leq p \leq \infty$) under the following restrictions:

$$\sum_{j=-\infty}^{\infty} |a_j| < \infty, \quad \int_{-\infty}^{\infty} |k(t)| dt < \infty. \quad (2)$$

The results obtained in this note generalize the results of M. G. Krein ⁽¹⁾ on Wiener-Hopf equations.

Let us also note that, under the restrictions (2), equation (1) can be written in the form

$$\int_{-\infty}^t \varphi(t-s) d\omega(s) = f(t) \quad (0 \leq t < \infty),$$

where $\omega(t)$ ($-\infty < t < \infty$) is a function of bounded variation without singular component (see (2)).

2. Denote by $\mathfrak{A}$ the set of all operators W acting in the space $L_p(0, \infty)$ by the formula

$$(W\varphi)(t) = \sum_{j=-\infty}^{\infty} a_j \varphi(t - \delta_j) + \int_0^{\infty} k(t-s) \varphi(s) ds, \quad (3)$$

where the numbers a_j and the function $k(t)$ satisfy conditions (2).

To the operator $W \in \mathfrak{A}$, defined by equality (3), we associate the function

$$\mathcal{W}(\lambda) = \sum_{j=-\infty}^{\infty} a_j e^{i\delta_j \lambda} + \int_{-\infty}^{\infty} k(t) e^{i\lambda t} dt \quad (-\infty < \lambda < \infty). \quad (4)$$

which we shall call the **symbol** of the operator W . We shall call the symbol $\mathcal{W}(\lambda)$ **nondegenerate** if

$$\inf |\mathcal{W}(\lambda)| > 0.$$

It is easy to see that every operator $W \in \widetilde{\mathfrak{A}}$ is a linear bounded operator in any of the spaces $L_p(0, \infty)$ ($1 \leq p \leq \infty$).

Theorem 1. *In order that an operator $W \in \widetilde{\mathfrak{A}}$ be invertible in the space $L_p(0, \infty)$ ($1 \leq p \leq \infty$) at least on one side, it is necessary and sufficient that its symbol $\mathcal{W}(\lambda)$ be nondegenerate.*

*If the latter condition is not satisfied, then the operator W is neither a Φ - nor a $\Phi_{\pm}$ -operator.**

3. Denote by $\mathfrak{A}$ the algebra of all functions $\mathcal{W}(\lambda)$ of the form (4) satisfying condition (2).

By $\mathfrak{A}_+(\mathfrak{A}_-)$ we denote the subalgebra of the algebra $\mathfrak{A}$ consisting of all functions of the form (4) for which $\delta_j \geq 0$ and $k(t) = 0$ for $t \leq 0$ ($\delta_j \leq 0$ and $k(t) = 0$ for $t \geq 0$).

The algebra $\mathfrak{A}$ is the direct sum of the subalgebra $\mathfrak{B}$ of almost periodic functions of the form $\sum a_j \exp(i\delta_j \lambda)$ ($\sum |a_j| < \infty$) and the subalgebra $\mathfrak{C}$ of Fourier transforms of functions from $L_1(-\infty, \infty)$. The subalgebra $\mathfrak{C}$ is an ideal of the algebra $\mathfrak{A}$.

If the function $\mathcal{W}(\lambda) = a(\lambda) + \mathcal{K}(\lambda)$ ($a(\lambda) \in \mathfrak{B}$, $\mathcal{K}(\lambda) \in \mathfrak{C}$) is nondegenerate, then its almost periodic component $a(\lambda)$ is also nondegenerate. With every nondegenerate function $\mathcal{W}(\lambda) \in \mathfrak{A}$ are associated two numbers $\nu(\mathcal{W})$ and $n(\mathcal{W})$, defined by the equalities

$$\nu(\mathcal{W}) = \lim_{l \rightarrow \infty} \frac{1}{2l} [\arg a(\lambda)]_{-l}^l, \quad n(\mathcal{W}) = \frac{1}{2\pi} [\arg(1 + a^{-1}(\lambda)\mathcal{K}(\lambda))]_{-\infty}^{\infty}.$$

The limit in the first equality exists by virtue of a known property of almost periodic functions (see (4)). The real number $\nu(\mathcal{W})$ and the integer $n(\mathcal{W})$ will be called the **indices** of the function $\mathcal{W}(\lambda)$.

Theorem 2. *Every nondegenerate function $\mathcal{W}(\lambda) \in \mathfrak{A}$ admits a factorization of the form*

$$\mathcal{W}(\lambda) = \mathcal{W}_-(\lambda) e^{i\lambda\nu} \left(\frac{\lambda - i}{\lambda + i} \right)^n \mathcal{W}_+(\lambda) \quad (-\infty < \lambda < \infty), \quad (5)$$

where $\mathcal{W}_\pm(\lambda), \mathcal{W}_\pm^{-1}(\lambda) \in \mathfrak{A}_\pm$; $\nu = \nu(\mathcal{W})$ and $n = n(\mathcal{W})$.

This theorem plays an important role in the proof of the sufficiency of the conditions of Theorem 1, and also in the proof of the following Theorems 3–6. The proof of Theorem 2 is based on a theorem of G. E. Shilov (see (2)) on analytic functions of elements of a commutative Banach algebra.

4. For the formulation of the subsequent theorems we shall need the operator $V^{(n)}$ ($n = 0, \pm 1, \dots$), whose symbol is equal to $(\lambda - i)^n / (\lambda + i)^n$. It is easy to see that $V^{(n)} = V^n$ ($n = 1, 2, \dots$) and $V^{(n)} = (V^{-1})^{-n}$ ($n = -1, -2, \dots$), where V and V^{-1} are the operators defined by the equalities

$$(V\varphi)(t) = \varphi(t) - 2 \int_0^t e^{s-t} \varphi(s) ds, \quad (V^{-1}\varphi)(t) = \varphi(t) - 2 \int_0^\infty e^{t-s} \varphi(s) ds.$$

Let us also note that if an operator $W(\in \tilde{\mathfrak{A}})$ is such that both functions $\mathcal{W}^{\pm 1}(\lambda) \in \mathfrak{A}_+$ ($\mathcal{W}^{\pm 1}(\lambda) \in \mathfrak{A}_-$), then it is invertible, with $W^{-1} \in \tilde{\mathfrak{A}}$, and the symbol of the operator W^{-1} is equal to $1/\mathcal{W}(\lambda)$.

In all subsequent theorems we shall assume that $W(\in \tilde{\mathfrak{A}})$ is an operator with nondegenerate symbol $\mathcal{W}(\lambda)$ and that equality (5) gives a factorization of the symbol $\mathcal{W}(\lambda)$. For brevity the numbers $\nu(\mathcal{W})$ and $n(\mathcal{W})$ will be denoted by ν and n . By $W_\pm$ is denoted the invertible operator with symbol $\mathcal{W}_\pm(\lambda)$.

Theorem 3. *If $\nu > 0$, then the operator W is left-invertible. In order that the equation $W\varphi = g$ be solvable, it is necessary and sufficient that:*

* For the definition of Φ - and $\Phi_\pm$ -operators see (3).

a) for $n \geq 0$ the function $W_-^{-1}g$ vanishes on the interval $[0, \nu]$ and the condition

$$\int_0^\infty (W_-^{-1}g)(t) t^k e^{-t} dt = 0 \quad (k = 0, 1, \dots, n-1); \quad (6)$$

is satisfied;

b) for $n < 0$ the function $V^{-n}W_-^{-1}g$ coincides on the interval $[0, \nu]$ with a function of the form

$$\sum_{j=0}^{-n-1} c_j t^j e^{-t},$$

where the c_j are complex numbers.

Theorem 4. If $\nu < 0$, then the operator W is right-invertible. Every solution $\varphi \in L_p(0, \infty)$ of the equation $W\varphi = 0$:

a) for $n \geq 0$ has the form

$$\varphi = W_+^{-1} V^{-n} g,$$

where g is an arbitrary function from $L_p(0, \infty)$ satisfying the conditions

$$g(t) = 0 \quad (t > -\nu); \quad \int_0^\infty g(t) t^j e^{-t} dt = 0 \quad (j = 0, 1, \dots, n-1);$$

b) for $n < 0$ has the form

$$\varphi = W_+^{-1} \left(g(t) + \sum_{j=0}^{-n-1} c_j t^j e^{-t} \right),$$

where g is an arbitrary function from L_p , vanishing on the interval $[-\nu, \infty]$, and the c_j are arbitrary complex numbers.

Theorem 5. If $\nu = 0$, then:

- a) for $n = 0$ the operator W is invertible;
- b) for $n > 0$ the operator W is left-invertible; the equation $W\varphi = g$ is solvable if and only if condition (6) is fulfilled;
- c) for $n < 0$ the operator W is right-invertible and the general solution of the equation $W\varphi = 0$ is given by the equality

$$\varphi = W_+^{-1} \left(\sum_{j=0}^{-n-1} c_j t^j e^{-t} \right),$$

where the c_j are arbitrary complex numbers.

5. Denote by U_ν the operator with symbol $e^{i\nu\lambda}$, i.e., the operator defined by the equalities $(U_\nu f)(t) = f(t-\nu)$ for $\max(\nu, 0) < t < \infty$, and $(U_\nu f)(t) = 0$ for $0 \leq t \leq \max(\nu, 0)$.

It is easy to see that if the numbers ν and n are positive, then the operator $U_\nu V^{(n)}$ is only left-invertible, while the operator $U_{-\nu} V^{(-n)}$ is only right-invertible. In the case $\nu > 0$ and $n < 0$ the equality $U_{-\nu} V^{(-n)} V^{(n)} U_\nu = I - U_{-\nu} P_{-n} U_\nu$ holds, where $P_{-n} = I - V^{(-n)} V^{(n)}$ is a finite-dimensional projector. It is readily proved that the operator $I - U_{-\nu} P_{-n} U_\nu$ is invertible in every L_p ($1 \leq p \leq \infty$).

The operator $W^{(-1)}$, inverse to the operator W from the corresponding side, is defined by the equalities

$$W^{(-1)} = W_+^{-1} V^{(-n)} U_{-\nu} W_-^{-1} \quad (\nu \geq 0, n \geq 0);$$

$$W^{(-1)} = W_+^{-1} (I - U_{-\nu} P_{-n} U_\nu)^{-1} U_{-\nu} V^{(-n)} W_-^{-1} \quad (\nu \geq 0, n \leq 0);$$

$$W^{(-1)} = W_+^{-1} V^{(-n)} U_{-\nu} W_-^{-1} \quad (\nu \leq 0, n \leq 0);$$

$$W^{(-1)} = W_+^{-1} V^{(-n)} U_{-\nu} (I - U_\nu P_n U_{-\nu})^{-1} W_-^{-1} \quad (\nu \leq 0, n \geq 0).$$

In the first two cases the operator $W^{(-1)}$ is a left inverse to the operator W , and in the last two cases the operator $W^{(-1)}$ is a right inverse to W . If at least one of the indices ν, n is nonzero, then the operator $W^{(-1)}$ is an inverse to W only from one side.

6. The restrictions on the symbol $\mathcal{W}(\lambda)$ can in some cases be weakened. For example, for the Hilbert space $L_2(0, \infty)$ the results listed above extend to the case of an operator W whose symbol $\mathcal{W}(\lambda) = a(\lambda) + \mathcal{K}(\lambda)$ is such that $a(\lambda)$ is an arbitrary almost periodic function, and $\mathcal{K}(\lambda)$ ($-\infty \leq \lambda \leq \infty$) is an arbitrary continuous function.

The results listed above extend to the corresponding adjoint equations and to their transposes.

7. Denote by P_τ ($0 < \tau < \infty$) the projector defined in the space $L_p(0, \infty)$ ($1 \leq p < \infty$) by the equality

$$(P_\tau \varphi)(t) = \begin{cases} \varphi(t), & t \leq \tau, \\ 0, & t > \tau. \end{cases}$$

If the operator $P_\tau W P_\tau$ is invertible in the subspace $P_\tau L_p(0, \infty)$, then by $(P_\tau W P_\tau)^{-1}$ we shall denote the operator equal to the inverse of the operator $P_\tau W P_\tau$ on the subspace $P_\tau L_p(0, \infty)$ and equal to zero on the subspace $(I - P_\tau) L_p(0, \infty)$. If the operator $P_\tau W P_\tau$ is not invertible, then we put $\|(P_\tau W P_\tau)^{-1}\| = \infty$.

Theorem 6. Let $W \in \tilde{\mathfrak{A}}$. If

$$\lim_{\tau \rightarrow \infty} \|(P_\tau W P_\tau)^{-1}\| < \infty,$$

then the symbol of the operator W is nondegenerate and $v(\mathcal{W}) = n(\mathcal{W}) = 0$.

When the last conditions are fulfilled, the equation

$$\sum_{t-\tau \leq \delta_j < t} a_j \varphi(t - \delta_j) + \int_0^\tau k(t-s) \varphi(s) ds = f(t) \quad (0 \leq t \leq \tau),$$

beginning with some τ , has a unique solution $\varphi_\tau(t) \in P_\tau L_p(0, \tau)$, and the functions

$$\tilde{\varphi}_\tau(t) = \begin{cases} \varphi_\tau(t), & t \leq \tau, \\ 0, & t > \tau. \end{cases}$$

converge as $\tau \rightarrow \infty$, in the norm of the space $L_p(0, \infty)$, to the solution of equation (1), whatever the function $f(t) \in L_p(0, \infty)$.

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Note: Figure translations are in progress. See original paper for figures.

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