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Abstract

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UDC 517

MATHEMATICS

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ON THE ORDER OF APPROXIMATION OF A FUNCTION BY ZYGMUND NORMAL MEANS

(Presented by Academician L. V. Kantorovich on 23 X 1967)

Let L_p ($1 \leq p \leq \infty$) denote the space of measurable 2π -periodic functions for which

$$\|f(x)\|_{L_p} = \left\{ \int_0^{2\pi} |f(x)|^p dx \right\}^{1/p} < \infty \quad (1 \leq p < \infty),$$

and, for $p = \infty$,

$$\operatorname{vraisup}_x |f(x)| < \infty.$$

For each function $f(x) \in L_p$ with Fourier series

$$\sum_{\nu=0}^{\infty} A_{\nu}(x), \quad \left(A_0(x) \equiv \frac{a_0}{2}, \quad A_{\nu}(x) = a_{\nu} \cos \nu x + b_{\nu} \sin \nu x, \quad \nu = 1, 2, \dots \right),$$

for any natural k we consider the operator

$$Z_n^{(k)}(f; x) = \sum_{\nu=0}^n \left(1 - \frac{\nu^k}{(n+1)^k} \right) A_{\nu}(x),$$

which is usually called the Zygmund normal mean of order k .

Numerous works have been devoted to the investigation of the order of approximation of each individual function $f(x) \in L_p$ by the operators $Z_n^{(k)}(f; x)$, depending on its structural properties. At the same time, as Zygmund already showed in ⁽¹⁾, there is a substantial difference in considering this question for the cases of even and odd k .

If k is an even number, and the function $f(x)$ has a $(k-1)$ -st absolutely continuous derivative and $\|f^{(k)}(x)\|_{L_{\infty}} < \infty$, then (see ⁽¹⁾)

$$\|f(x) - Z_n^{(k)}(f; x)\|_{L_{\infty}} \leq c_k (n+1)^{-k} \|f^{(k)}(x)\|_{L_{\infty}}. \quad (1)$$

Another proof of this Zygmund inequality was given by the author in ⁽⁸⁾. It is based on the obvious estimate

$$\|f(x) - Z_n^{(k)}(f; x)\|_{L_p} \leq (1 + \|Z_n^{(k)}\|) \|f(x) - T_n(x)\|_{L_p} + (n+1)^{-k} \left\| \frac{d^k}{dx^k} T_n(x) \right\|_{L_p} \quad (2)$$

valid for any trigonometric polynomial $T_n(x)$ (see ⁽⁸⁾ for $p = \infty$ and ⁽⁷⁾, p. 591, for any $1 \leq p \leq \infty$).

From inequality (1) and one result of S. B. Stechkin ⁽⁴⁾ it follows that, for any continuous function $f(x)$,

$$\|f(x) - Z_n^{(k)}(f; x)\|_{L_\infty} \leq c_k \omega_k \left(f; \frac{1}{n+1} \right)_{L_\infty}, \quad (3)$$

where

$$\omega_k(f; h)_{L_p} = \sup_{|t| \leq h} \|\Delta_t^k f(x)\|_{L_p} = \sup_{|t| \leq h} \left\| \sum_{\nu=0}^k (-1)^{k-\nu} \binom{k}{\nu} f(x + \nu t) \right\|_{L_p}.$$

Estimate (3) and the analogous estimate for the case where $f(x) \in L_p$ ($1 \leq p \leq \infty$),

$$\|f(x) - Z_n^{(k)}(f; x)\|_{L_p} \leq c_k \omega_k \left(f; \frac{1}{n+1} \right)_{L_p}, \quad (4)$$

follow from (2), if one uses the following inequality for derivatives of a trigonometric polynomial (see ^(5,6)):

$$\left\| \frac{d^k}{dx^k} T_n(x) \right\|_{L_p} \ll \left(\frac{n}{2} \right)^k \|\Delta_{\pi/2n}^k T_n(x)\|_{L_p}. \quad (5)$$

R. M. Trigub ⁽¹⁰⁾ and V. V. Zhuk ⁽¹¹⁾ independently showed that, for even k , for any function $f(x) \in L_p$ the converse to inequality (4) is also true, i.e.,

$$\omega_k \left(f; \frac{1}{n+1} \right)_{L_p} \ll c_k \|f(x) - Z_n^{(k)}(f; y)\|_{L_p} \quad (1 \leq p \leq \infty). \quad (6)$$

Thus, for any function $f(x) \in L_p$ ($1 \leq p \leq \infty$), for even k the order relation $*$ holds:

$$\|f(x) - Z_n^{(k)}(f; x)\|_{L_p} \asymp \omega_k \left(f; \frac{1}{n+1} \right)_{L_p}. \quad (7)$$

In the case when k is an odd number and $1 < p < \infty$, relation (7) also remains valid. Estimate (4) in this case for $k = 1$ was obtained by P. L. Ul'yanov ⁽⁹⁾,

and for other values of k by Yu. A. Ponomarenko ⁽¹²⁾. Inequality (6) can be obtained with the aid of the identity

$$\frac{d^k}{dx^k} Z_n^{(k)}(f; x) = (-1)^{k/2} (n+1)^k \left\{ \tilde{Z}_n^{(k)}(f; x) - \tilde{Z}_{n,1}^{(k)}(f; x) \right\}, \quad (8)$$

where $\tilde{Z}_n^{(k)}(f; x)$ are the Zygmund normal means for the series conjugate to the Fourier series of the function $f(x)$, and $\tilde{Z}_{n,1}^{(k)}(f; x) = Z_n^{(k)}(\tilde{Z}_n^{(k)}(f); x)$. Indeed, for any h ($0 < h \leq \frac{1}{n+1}$),

$$\begin{aligned} \|\Delta_h^k f(x)\|_{L_p} &\leq \|\Delta_h^k f(x) - \Delta_h^k Z_n^{(k)}(f; x)\|_{L_p} + h^k \left\| \frac{d^k}{dx^k} Z_n^{(k)}(f; x) \right\|_{L_p} \leq \\ &\leq 2^k \|f(x) - Z_n^{(k)}(f; x)\|_{L_p} + \|\tilde{Z}_n^{(k)}(f; x) - \tilde{Z}_{n,1}^{(k)}(f; x)\|_{L_p}. \end{aligned}$$

Since $1 < p < \infty$, the function $\tilde{f}(x)$, conjugate to $f(x)$, also belongs to the space L_p , and, by the well-known inequality of M. Riesz (see ⁽¹³⁾, p. 149), we find

$$\begin{aligned} \|\Delta_h^k f(x)\|_{L_p} &\leq 2^k \|f(x) - Z_n^{(k)}(f; x)\|_{L_p} + \|\tilde{f}(x) - \tilde{Z}_n^{(k)}(f; x)\|_{L_p} + \\ &+ \|\tilde{f}(x) - \tilde{Z}_{n,1}^{(k)}(f; x)\|_{L_p} \leq c_{p,k} \left\{ \|f(x) - Z_n^{(k)}(f; x)\|_{L_p} + \|f(x) - Z_{n,1}^{(k)}(f; x)\|_{L_p} \right\}, \end{aligned}$$

where

$$Z_{n,1}^{(k)}(f; x) = Z_n^{(k)}(Z_n^{(k)}(f); x) = \sum_{\nu=0}^n \left(1 - \frac{\nu^k}{(n+1)^k} \right)^2 A_\nu(x).$$

From this it is easy to obtain

$$\|\Delta_h^k f(x)\|_{L_p} \ll M_{p,k} \|f(x) - Z_n^{(k)}(f; x)\|_{L_p}.$$

From this estimate we obtain (6), with a constant depending not only on k , but also on p .

For $p = \infty$ or $p = 1$ and odd k , the order relation (7), as is known, does not hold. In these cases, when studying the order of approximation of a function $f(x)$ by the operators $Z_n^{(k)}(f; x)$, as has been shown in a number of works (see, for example, ⁽¹⁻⁴⁾), it is important to take into account the structural properties

not only of the function $f(x)$ itself, but also the properties of the function $\tilde{f}(x)$ conjugate to it.

Below the following assertion is established concerning the approximation of a function $f(x) \in L_p$ ($1 \leq p \leq \infty$) by the Zygmund normal means $Z_n^{(k)}(f; x)$ for the case when k is any odd number.

* The relation $u \asymp v$ means that $c_1 v \leq u \leq c_2 v$, where $c_1 > 0$, $c_2 > 0$ are some constants.

Theorem. If $f(x) \in L_p$ ($1 \leq p \leq \infty$) has Fourier series $\sum_{\nu=0}^{\infty} A_{\nu}(x)$, then for any odd k the following order relation holds:

$$\|f(x) - Z_n^{(k)}(f; x)\|_{L_p} \asymp \omega_{k+1}\left(f; \frac{1}{n+1}\right)_{L_p} + \omega_{k+1}\left(\tilde{F}; \frac{1}{n+1}\right)_{L_p} (n+1), \quad (9)$$

where $\tilde{F}(x)$ is a function having Fourier series $\sum_{\nu=1}^{\infty} \frac{1}{\nu} A_{\nu}(x)$.*

The proof of the theorem is based on the following, easily verified identities:**

$$(n+1)^{-k-1} \frac{d^{k+1}}{dx^{k+1}} Z_n^{(k)}(f; x) = (-1)^{(k-1)/2} \left[Z_n^{(k)}(f; x) - Z_{n,1}^{(k)}(f; x) - Z_n^{(k)}(Z_n^{(1)}(f); x) + Z_{n,1}^{(k)}(Z_n^{(1)}(f); x) \right], \quad (10)$$

$$(n+1)^{-k} \frac{d^{k+1}}{dx^{k+1}} Z_n^{(k+1)}(\tilde{F}; x) = (-1)^{(k-1)/2} \left[Z_n^{(k+1)}(f; x) - Z_n^{(k+1)}(Z_n^{(k)}(f); x) \right], \quad (11)$$

$$Z_n^{(k+1)}(f; x) = Z_n^{(k)}(f; x) + Z_n^{(1)}(f; x) - Z_n^{(1)}(Z_n^{(k)}(f); x). \quad (12)$$

Let us first establish that for any p ($1 \leq p \leq \infty$) and odd k

$$\|f(x) - Z_n^{(k)}(f; x)\|_{L_p} \asymp c_k \left\{ \omega_{k+1}\left(f; \frac{1}{n+1}\right)_{L_p} + (n+1) \omega_{k+1}\left(\tilde{F}; \frac{1}{n+1}\right)_{L_p} \right\}. \quad (13)$$

Obviously,

$$\begin{aligned} \|f(x) - Z_n^{(k)}(f; x)\|_{L_p} &\leq \|f(x) - Z_n^{(k+1)}(f; x)\|_{L_p} + \|Z_n^{(k)}(f - Z_n^{(k+1)}(f); x)\|_{L_p} \\ &\quad + \|Z_n^{(k+1)}(f; x) - Z_n^{(k)}(Z_n^{(k+1)}(f); x)\|_{L_p} = U_1 + U_2 + U_3. \end{aligned} \quad (14)$$

Since $k+1$ is an even number, by (4) and the boundedness of the norm of the operator $Z_n^{(k)}(f; x)$, we find

$$U_1 \preccurlyeq c_k \omega_{k+1} \left(f; \frac{1}{n+1} \right)_{L_p}, \quad U_2 \preccurlyeq b_k \omega_{k+1} \left(f; \frac{1}{n+1} \right)_{L_p}. \quad (15)$$

Using identity (11) and inequalities (5) and (4), we obtain

$$\begin{aligned} U_3 &\preccurlyeq (n+1)^{-k} \left\| \frac{d^{k+1}}{dx^{k+1}} Z_n^{(k+1)}(\tilde{F}; x) \right\|_{L_p} \\ &\preccurlyeq B_k(n+1) \omega_{k+1} \left(Z_n^{(k+1)}(\tilde{F}); \frac{1}{n+1} \right)_{L_p} \\ &\preccurlyeq M_k(n+1) \left\{ \omega_{k+1} \left(\tilde{F}; \frac{1}{n+1} \right)_{L_p} + \left\| \tilde{F}(x) - Z_n^{(k+1)}(\tilde{F}; x) \right\|_{L_p} \right\} \\ &\preccurlyeq M'_k(n+1) \omega_{k+1} \left(\tilde{F}; \frac{1}{n+1} \right)_{L_p}. \end{aligned} \quad (16)$$

From estimates (15), (16), and (14), inequality (13) follows.

We shall now show that for any odd k the inequalities

$$\omega_{k+1} \left(f; \frac{1}{n+1} \right)_{L_p} \preccurlyeq c_k \|f(x) - Z_n^{(k)}(f; x)\|_{L_p}, \quad (17)$$

$$\omega_{k+1} \left(\tilde{F}; \frac{1}{n+1} \right)_{L_p} \preccurlyeq \frac{c_k}{n+1} \|f(x) - Z_n^{(k)}(f; x)\|_{L_p}. \quad (18)$$

* We note that in the special case when $k=1$ (Fejér sum), relation (9) was announced, without indication of its proof, by V. V. Zhuk at the interuniversity summer scientific school on summability theory, Sverdlovsk, July 1967.

** Such identities, with an indication of their application to estimates of deviations of functions from their normal Zygmund means, were earlier presented by the author in a report at the interuniversity seminar on function theory, Dnepropetrovsk, April 21, 1967, and at the summer scientific school, Sverdlovsk, July 1967.

Let $0 < h \leq \frac{1}{n+1}$. Using identity (10), we find

$$\begin{aligned}
 \|\Delta_h^{k+1} f(x)\|_{L_p} &\leq 2^{k+1} \|f(x) - Z_n^k(f; x)\|_{L_p} + \|\Delta_h^{k+1} Z_n^{(k)}(f; x)\|_{L_p} \\
 &\leq 2^{k+1} \|f(x) - Z_n^{(k)}(f; x)\|_{L_p} + h^{k+1} \left\| \frac{d^{k+1}}{dx^{k+1}} Z_n^{(k)}(f; x) \right\|_{L_p} \\
 &\leq 2^{k+1} \|f(x) - Z_n^{(k)}(f; x)\|_{L_p} \\
 &\quad + \|Z_n^{(k)}(f; x) - Z_{n,1}^{(k)}(f; x) - Z_n^k(Z_n^{(1)}(f); x) + Z_{n,1}^{(k)}(Z_n^{(1)}(f); x)\|_{L_p}.
 \end{aligned} \tag{19}$$

Adding and subtracting under the norm sign the difference $f(x) - Z_n^{(1)}(f; x)$, we obtain

$$\begin{aligned}
 &\|Z_n^{(k)}(f; x) - Z_{n,1}^{(k)}(f; x) - Z_n^k(Z_n^{(1)}(f); x) + Z_{n,1}^{(k)}(Z_n^{(1)}(f); x)\|_{L_p} \\
 &\leq \|f(x) - Z_n^{(k)}(f; x)\|_{L_p} + \|f(x) - Z_{n,1}^{(k)}(f; x)\|_{L_p} \\
 &\quad + \|Z_n^{(k)}(f - Z_n^{(k)}(f); x)\|_{L_p} + \|Z_n^{(1)}(f - Z_n^{(k)}(f); x)\|_{L_p}.
 \end{aligned} \tag{20}$$

Since, obviously,

$$\|f(x) - Z_{n,1}^{(k)}(f; x)\|_{L_p} \leq B_k \|f(x) - Z_n^{(k)}(f; x)\|_{L_p},$$

it follows from (19) and (20) that we arrive at the estimate

$$\|\Delta_h^{k+1} f(x)\|_{L_p} \leq c_k \|f(x) - Z_n^{(k)}(f; x)\|_{L_p}.$$

In view of the fact that this inequality is valid for every $0 < h \leq \frac{1}{n+1}$, we obtain estimate (17). Similarly, using identity (11) and identity (12) as applied to the function $\tilde{F}(x)$, for every $h \left(0 < h \leq \frac{1}{n+1}\right)$,

$$\begin{aligned}
 \|\Delta_h^{k+1} \tilde{F}(x)\|_{L_p} &\leq 2^{k+1} \|\tilde{F}(x) - Z_n^{(k+1)}(\tilde{F}; x)\|_{L_p} + h^{k+1} \left\| \frac{d^{k+1}}{dx^{k+1}} Z_n^{(k+1)}(\tilde{F}; x) \right\|_{L_p} \\
 &\leq 2^{k+1} \|\tilde{F}(x) - Z_n^{(k)}(\tilde{F}; x) + Z_n^{(1)}(\tilde{F}; x) - Z_n^{(1)}(Z_n^{(k)}(\tilde{F}); x)\|_{L_p} \\
 &\quad + \frac{1}{n+1} \|Z_n^{(k+1)}(f; x) - Z_n^{(k+1)}(Z_n^{(k)}(f); x)\|_{L_p} \\
 &\leq 2^{k+1} \|\varphi_k(x) - Z_n^{(1)}(\varphi_k; x)\|_{L_p} + \frac{1}{n+1} \|Z_n^{(k+1)}(f - Z_n^{(k)}(f); x)\|_{L_p},
 \end{aligned}$$

where $\varphi_k(x) = \tilde{F}(x) - Z_n^{(k)}(\tilde{F}; x)$. Applying to the function $\varphi_k(x)$ the known theorem of Alexits (3), we obtain that

$$\|\varphi_k(x) - Z_n^{(1)}(\varphi_k; x)\|_{L_p} \leq C \frac{\|\tilde{\varphi}_k(x)\|_{L_p}}{n+1} = C \frac{\|f(x) - Z_n^{(k)}(f; x)\|_{L_p}}{n+1},$$

where $\tilde{\varphi}_k(x)$ is the function conjugate to $\varphi_k(x)$. Thus,

$$\|\Delta_h^{k+1}\tilde{F}(x)\|_{L_p} \leq \frac{c_k}{n+1} \|f(x) - Z_n^{(k)}(f; x)\|_{L_p}.$$

From this inequality (18) follows. From the estimates (17), (18), and (13) we arrive at the order relation (9).

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Note: Figure translations are in progress. See original paper for figures.

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