

ON HAAR SERIES CONVERGING ALONG SUBSEQUENCES OF PARTIAL SUMS

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Abstract

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MATHEMATICS

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ON HAAR SERIES CONVERGING ALONG SUBSEQUENCES OF PARTIAL SUMS

(Presented by Academician A. N. Kolmogorov on 11 III 1968)

In paper ⁽¹⁾ it was shown that every everywhere convergent series with respect to the Haar system may be regarded as a Fourier series for its sum, if in the Fourier formulas one uses the *HD*-integral, which is a certain generalization of the Denjoy integral.

In the present note it is shown how an analogous problem can be solved for a series with respect to the Haar system that converges only along subsequences of partial sums. Further, the properties of the class of generalized integrals introduced for the solution of this problem are considered, as well as the connection of these integrals with known integrals, in particular with the broad Denjoy integral.

Consider a series with respect to the Haar system

$$\sum_{n=1}^{\infty} a_n \chi_n(x). \quad (1)$$

Put $\psi_n(x) = \int_0^x a_n \chi_n(t) dt$. The sum of the series

$$\sum_{n=1}^{\infty} \psi_n(x) \quad (2)$$

at points of convergence will be denoted by $\psi(x)$. It is easy to see that the sum $\psi(x)$ is defined at every dyadic-rational point x for any series (1).

We shall denote by R the set of dyadic-rational points on $[0, 1]$, and by I the dyadic-irrational ones. Points of R of the form $i/2^j$, where $i = 1, 2, \dots, 2^j$, we shall call elements of the net R_j of order j ($j = 0, 1, 2, \dots$). In the definition of the Haar system there occur intervals of the form

$$((i-1)/2^j, i/2^j), \quad j = 0, 1, 2, \dots; \quad i = 1, 2, \dots, 2^j. \quad (3)$$

An interval of the form (3), outside which the function $\chi_n(x)$ is equal to zero, will be called the support of the function $\chi_n(x)$. To each point $x \in I$ there corresponds a unique sequence of intervals of the form (3), for each of which the point x is an interior point. We denote this sequence by $\{u_k\} = \{(a_k, b_k)\}$, $k = 0, 1, 2, \dots$. To a point $x \in R$ there correspond two sequences (left and right) of intervals of the form (3), for which the point x is a common endpoint.

Definition 1. Let $\{(a_k, b_k)\}$ be a sequence of intervals of the form (3) converging to a point $x \in I$, and let $\{k_i(x)\}$ be some infinite subsequence of natural numbers corresponding to this point. Then for a function $\psi(x)$, given at the endpoints of the intervals $\{(a_k, b_k)\}$, we define the **derivative with respect to the subsequence $\{k_i(x)\}$ of the nets R_k** , or the $R(k_i(x))$ -derivative, at the point x by the equality

$$R(k_i(x))D\psi(x) = \lim_{i \rightarrow \infty} \frac{\psi(b_{k_i}) - \psi(a_{k_i})}{b_{k_i} - a_{k_i}}.$$

Limits over subsequences of natural numbers i in this expression will be called $R(k_i(x))$ -derived numbers. Thus, in particular, the upper and lower $R(k_i(x))$ -derived numbers will be defined.

Definition 2. Let $x \in R$, and let $\{(a_k, b_k)\}$, $\{a'_k, b'_k\}$ be left and right sequences of intervals of the form (3) contracting to it, while $\{k_i(x)\}$ and $\{k'_i(x)\}$ are two corresponding (left and right) subsequences of natural numbers for this point. Then $\psi(x)$ is called $R(k_i(x), k'_i(x))$ -continuous at the point x if

$$\lim_{i \rightarrow \infty} \{(\psi(a_{k_i}) + \psi(b_{k_i}))/2 - \psi(a_{k_{i+1}})\} = 0, \quad \lim_{i \rightarrow \infty} \{(\psi(b'_{k'_i}) + \psi(a'_{k'_i}))/2 - \psi(b'_{k'_{i+1}})\} = 0.$$

Theorem 1. Let for each point $x \in I$ a subsequence of natural numbers $\{k_i(x)\}$ be defined, and for $x \in R$ let a pair of subsequences $\{k_i(x), k'_i(x)\}$ be defined; and let the function $\psi(x)$ be defined for all $x \in R$, be $R(k_i(x), k'_i(x))$ -continuous at every point $x \in R$, and, at every point $x \in I$, satisfy $\underline{R(k_i(x))}D\psi(x) \geq 0$. Then $\psi(x)$ does not decrease on R .

We can now give the following generalization of the Perron integral.

Definition 3. Let a finite function $f(x)$ be defined everywhere on $[0, 1]$. Then a function $M(x)$ is called an $R(k_i(x))P$ -majorant if:

1) $M(0) = 0$; 2) $\underline{R(k_i(x))}DM(x) \geq f(x)$ for all $x \in I$; 3) the function $M(x)$ is $R(k_i(x), k'_i(x))$ -continuous for all $x \in R$.

The $R(k_i(x))P$ -minorant $m(x)$ is defined analogously. By Theorem 1, the difference $M(x) - m(x)$ does not decrease on R ; in particular, $M(1) \geq m(1)$.

Definition 4. If $J = \inf M(1) = \sup m(1)$, where the bounds are taken over the set of all $R(k_i(x))P$ -majorants and minorants, then $f(x)$ is called $R(k_i(x))P$ -integrable on $[0, 1]$, and

$$J = R(k_i(x))P - \int_0^1 f(x) dx.$$

Here the indefinite integral is defined at least for all $x \in R$.

Let us apply the definition of integral introduced here to series with respect to the Haar system. First of all, note that series (1) at each point x actually reduces to the series

$$\sum_{k=0}^{\infty} a_{n_k} \chi_{n_k}(x), \quad (4)$$

where $\{n_k\}$ is the sequence of indices for which the Haar functions $\chi_n(x)$ are nonzero at the point x . Then, if series (1) converges at the point x along some subsequence of partial sums to the value $\varphi(x)$, then series (4) also converges to the same value along some subsequence of its partial sums with indices $\{k_i(x)\}$. We can now formulate the following theorem.

Theorem 2. Let to each point $x \in I$ there correspond a sequence of indices $\{k_i(x)\}$ along which the partial sums of series (4) converge to $\varphi(x)$, and to each point $x \in R$ there correspond a pair of subsequences $\{k_i(x), k'_i(x)\}$ such that

$$\lim_{i \rightarrow \infty} \frac{a_{n_{k_i}}}{\chi_{n_{k_i}}(x)} = 0, \quad \lim_{i \rightarrow \infty} \frac{a_{n_{k'_i}}}{\chi_{n_{k'_i}}(x)} = 0, \quad (5)$$

where $\{n_k\}$ and $\{n_{k'}\}$ are the sequences of indices for which the point x is respectively the right and left endpoint of the supports of the functions $\chi_n(x)$. Then $\varphi(x)$ is $R(k_i(x))P$ -integrable and the coefficients of series (4), and consequently also of series (1), are computed by the usual Fourier formulas.

Since the Perron integral is, evidently, covered by the $R(k_i(x))P$ -integral for arbitrary sequences $\{k_i(x)\}$, it follows from Theorem 2 that

Theorem 3. *If, under the hypotheses of Theorem 2, the function $\varphi(x)$ is P -integrable, then the series (1) is the Fourier-Perron series of the function $\varphi(x)$.*

This theorem generalizes the analogous theorem of F. G. Arutyunyan and A. A. Talalyan (see (2)) for L -integrable functions. As we shall see below, such a theorem is no longer true for functions $\varphi(x)$ integrable in the broad Denjoy sense (D -integrable).

We note the following property of the integrals introduced here.

Theorem 4. *The integrals $R(k_i(x))P$ and $R(l_i(x))P$ may contradict one another for different sequences $\{k_i(x)\}$ and $\{l_i(x)\}$.*

Let us observe that this contradiction is essential, since it persists on the class of sums convergent along subsequences of Haar series. Theorem 4 and a number of other assertions follow from the following example.

Example 1. There exists a D -integrable function $\varphi(x)$ whose Fourier-Denjoy series, represented in the form (4), converges along some subsequence $\{k_i(x)\}$ at every point $x \in I$, while some other series

$$\sum_{k=0}^{\infty} c_{n_k} \chi_{n_k}(x),$$

where c_{n_k} differ from a_{n_k} for some indices, converges for all $x \in I$ to $\varphi(x)$ along some other subsequence $\{l_i(x)\}$. Moreover, condition (5) is satisfied for both series (it is useful to note that, in this case, the sequences $\{k_i(x)\}$ and $\{l_i(x)\}$ may coincide for points x from some set of full measure).

In addition to Theorem 4, it also follows from Example 1 that the $R(k_i(x))P$ -integral may contradict the D -integral, and that Theorem 3 cannot be extended to the class of D -integrable functions.

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Note: Figure translations are in progress. See original paper for figures.

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