

ON GLOBAL SOLVABILITY OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

Yu. A. Dubinskii

1968

SovietRxiv

View the original and related papers at <https://sovietsrxiv.org/items/ru-196801.28080>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.91

MATHEMATICS

Yu. A. DUBINSKII

ON GLOBAL SOLVABILITY OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

(Presented by Academician I. G. Petrovskii on 18 XII 1967)

It is known that, in the case of an arbitrary function $f(t, u)$, the Cauchy problem for equation (1) is not globally solvable for every right-hand side $h(t)$, i.e., for every time $t \geq 0$. The question of how large the set of functions $h(t)$ is for which global solvability holds has, roughly speaking, the following answer: the codimension of this set in the space $\mathcal{L}_2(0, \infty)$ is equal to one. This fact is proved in § 1. In § 2 an analogous result is established for an equation of order n .

§ 1. Equation of the first order. Consider on the half-axis the Cauchy problem for the equation

$$u' + f(t, u) = h(t), \quad u(0) = 0. \quad (1)$$

Notation:

$${}^0H^{(n)}(0, T) = \left\{ u(t) \mid \int_0^T |u^{(n)}(t)|^2 dt < \infty, u(0) = u(T) = 0, \dots, u^{(n-1)}(0) = u^{(n-1)}(T) = 0 \right\};$$

$${}^0H^{(n)}(0, \infty) = \left\{ u(t) \mid \int_0^\infty |u^{(n)}(t)|^2 dt < \infty, u(0) = 0, \dots, u^{(n-1)}(0) = 0 \right\}.$$

Theorem 1. Let $f(t, \xi)$ be differentiable with respect to t and locally summable with respect to ξ , and suppose that for any finite function $u(t)$ on the half-axis

$$\int_0^u \dot{f}(t, u) u' dt \geq -K,$$

where $K \geq 0$ is a constant.

Then for any function $h(t) \in \mathcal{L}_2(0, \infty)$ there exist a constant C and a function $u(t) \in \overset{0}{H}^{(1)}(0, \infty)$ such that

$$u' + f(t, u) = h(t) + C. \quad (2)$$

In other words, equation (1) is globally solvable up to an additive constant.

Proof. Let $v_1(t), v_2(t), \dots$ be an orthonormal in $\mathcal{L}_2(0, \infty)$ sequence of smooth functions finite on the half-axis, complete in $\overset{0}{H}^{(1)}(0, T)$ for every finite T . (As such a sequence one may take, for example, the union of sequences complete in $\overset{0}{H}^{(1)}(0, T)$, where T is a natural number, and normalize it in $\mathcal{L}_2(0, \infty)$.)

We seek an approximate solution $u_n(t)$ in the form

$$u_n(t) = \sum_{k=1}^n c_{kn} v_k(t),$$

where the constants c_{kn} are determined from the system of equations

$$\int_0^\infty u'_n v'_k dt + \int_0^\infty f(t, u_n) v'_k dt = \int_0^\infty h v'_k dt, \quad k = 1, \dots, n. \quad (3)$$

The solvability of system (3) follows from a lemma of M. I. Vishik ⁽¹⁾; moreover, for the approximate solutions the a priori estimate holds

$$\int_0^\infty |u'_n|^2 dt \leq K \left(\int_0^\infty h^2 dt + 1 \right).$$

This means that there exists a function $u(t) \in \overset{0}{H}^{(1)}(0, \infty)$ such that some subsequence $u_s(t) \rightarrow u(t)$ weakly in $\overset{0}{H}^{(1)}(0, \infty)$. Since, at the same time, the functions $u_s(t)$ themselves converge to $u(t)$ uniformly on any finite interval $[0, T]$, it follows from (3) that for any finite function $v(t)$

$$\int_0^\infty (u' + f(t, u) - h) v' dt = 0,$$

which is equivalent to (2). The theorem is proved.

Remark 1. It is clear that the proof may be regarded as an actual process of finding an almost-solution $u(t)$.

Remark 2. It is also clear that if on some interval (a, b) $h(t) \in H^{(s)}(a, b)$ and $f(t, \xi)$ is sufficiently smooth, then $u(t) \in H^{(s+1)}(a, b)$. In particular, if $h(t) \in C^\infty(a, b)$, then $u(t) \in C^\infty(a, b)$.

Remark 3. Theorem 1 is valid for any initial condition $u(0) = u_0$. For the proof it suffices to put $w(t) = u(t) - u_0$ and to apply the theorem just proved to the equation $w' + f(t, w - u_0) = h$.

Corollary 1. Under the conditions of Theorem 1, for any $h(t) \in \mathcal{L}_2(0, T)$ the Cauchy problem (1) is almost solvable, i.e., there exist C and $u(t)$

$$\left(u(0) = 0, \int_0^T |u'|^2 dt < \infty \right)$$

such that (2) holds.

Indeed, let $h^*(t)$ be the extension by zero of the function $h(t)$ to the whole half-axis $[0, \infty)$. By Theorem 1 there exists an almost-solution $u^*(t) \in H^{(1)}(0, \infty)$. Then the restriction of $u^*(t)$ to the interval $[0, T]$ gives the answer.

The following theorem on the solvability of the boundary-value problem is also valid.

Theorem 2. Under the conditions of Theorem 1, for any $h(t) \in \mathcal{L}_2(0, T)$ there exist a constant C and a function $u(t) \in H^{(1)}(0, T)$ such that

$$u' + f(t, u) = h(t) + C.$$

§ 2. **The general case.** Consider on the half-axis $[0, \infty)$ the Cauchy problem

$$u^{(n)} + f(t, u, \dots, u^{(n-1)}) = h(t), \quad (4)$$

$$u(0) = 0, \dots, u^{(n-1)}(0) = 0. \quad (5)$$

Theorem 3. Let $f(t, \xi_0, \dots, \xi_{n-1})$ be locally summable in each variable, and for any finite function $u(t)$ on the half-axis

$$\int_0^\infty f(t, u, \dots, u^{(n-1)}) u^{(n)} dt \geq -K, \quad (6)$$

where $K \geq 0$ is a constant.

Then for any $h(t) \in \mathcal{L}_2(0, \infty)$ there exists a polynomial

$$P_{n-1}(t) = C_0 + C_1 t + \dots + C_{n-1} t^{n-1}$$

and a function $u(t) \in \overset{0}{H}^{(n)}(0, \infty)$ such that

$$u^{(n)} + f(t, u, \dots, u^{(n-1)}) = h(t) + P_{n-1}(t). \quad (7)$$

The proof of this theorem repeats the proof of Theorem 1.

Remark 4. Simple examples show that, if condition (6) is violated, Theorem 3, generally speaking, does not hold.

Corollary 2. Under the conditions of Theorem 3, for any $h(t) \in \mathcal{L}_2(0, T)$ there exist a polynomial $P_{n-1}(t)$ and a function $u(t)$ ($u(0) = 0, \dots, u^{(n-1)}(0) = 0$,

$$\int_0^T |u^{(n)}|^2 dt < \infty)$$

such that (7) holds.

Corollary 3. If on some interval (a, b) , $h(t) \in H^{(s)}(a, b)$ and $f(t, \xi_0, \dots, \xi_{n-1})$ is sufficiently smooth, then $u(t) \in H^{(s+n)}(a, b)$. In particular, for $h(t) \in C^\infty(a, b)$ the almost-solution $u(t) \in C^\infty(a, b)$.

Theorem 4 (boundary-value problem). Under the conditions of Theorem 3, for any $h(t) \in \mathcal{L}_2(0, T)$ there exists a polynomial $P_{n-1}(t)$ such that equation (7) is solvable in $\overset{0}{H}^{(n)}(0, T)$.

We give the simplest examples of equations for which the conditions of the preceding theorems are fulfilled.

1.

$$u' + a_0(u) = h(t).$$

2.

$$u'' + a_0(u') + a_1(u) = h(t).$$

3.

$$u^{(n)} + a_0(u^{(n-1)}) + a_1(u^{(n-2)}) + a_2(u^{(n-3)}) + \dots + a_{n-1}(u) = h(t).$$

Here the function $a_0(\xi)$ is arbitrary, $a_1(\xi)$ is a nonincreasing function, a_k ($k \geq 2$) are numbers, with $(-1)^m a_{2m} \geq 0$, while a_{2m+1} are arbitrary.

Moscow

Power Engineering Institute

Received

13 XII 1967

REFERENCES

1. M. I. Vishik, DAN, **137**, No. 3, 502 (1961).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.