

ON SPACES OF CLOSED SUBSETS

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Abstract

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MATHEMATICS

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ON SPACES OF CLOSED SUBSETS

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Let X be a topological space. Denote by $\exp X$ the new topological space whose points are the nonempty closed subsets F of the space X , and whose open base consists of the sets

$$\langle U_1, \dots, U_n \rangle = \mathcal{E} \left(F \in \exp X : F \subseteq \bigcup_{k=1}^n U_k, F \cap U_k \neq \emptyset, k = 1, 2, \dots, n \right),$$

where $U_1, \dots, U_n$ is an arbitrary finite collection of nonempty open sets of the space X . Numerous properties of the space $\exp X$ are described in detail in ⁽²⁻⁵⁾. It should be noted that the space $\exp X$, endowed with the indicated topology, is always a T_0 -space. If, however, X is normal, then $\exp X$ is completely regular ⁽²⁾. Moreover, the conditions that X is (bi)compact and that $\exp X$ is (bi)compact are equivalent ⁽³⁾.

In this note the connection between the weight characteristics of the spaces X and $\exp X$ is investigated, a topological description of $\exp X$ is given for certain spaces X , and, finally, the connection between the dimensions of X and $\exp X$ is studied.

Lemma 1. *The system*

$$\mathfrak{B} = \{ \langle U_{\alpha_1}, \dots, U_{\alpha_n} \rangle \},$$

whose elements are all possible nonempty finite collections of sets of the open base

$$\mathfrak{b} = \{ U_\alpha \}$$

of the bicom pactum X , forms an open base of the bicom pactum $\exp X$.

Proof. Let $F \in \langle G_1, \dots, G_n \rangle \subseteq \exp X$. Since $G_1, \dots, G_n$ are open in X , and $\mathfrak{b}$ is a base in X , we have

$$G_k = \bigcup_{\xi} U_{\alpha(k)}^{\xi}, \quad k = 1, 2, \dots, n.$$

From the definition of the topology in $\exp X$ it follows that in X

$$F \subseteq \bigcup_{k=1}^n \bigcup_{\xi} U_{\alpha(k)}^{\xi}; \quad (1)$$

$$F \cap G_k = F \cap \left(\bigcup_{\xi} U_{\alpha(k)}^{\xi} \right) = \bigcup_{\xi} (F \cap U_{\alpha(k)}^{\xi}) \neq \emptyset, \quad k = 1, 2, \dots, n. \quad (2)$$

For each $k = 1, 2, \dots, n$ choose from the covering $\{U_{\alpha(k)}^{\xi}\}$ of the bicom pactum $F \subseteq X$ those $U_{\alpha(k)}^{\xi}$ for which $F \cap U_{\alpha(k)}^{\xi} \neq \emptyset$. As (2) shows, such $U_{\alpha(k)}^{\xi}$ exist for every $k = 1, 2, \dots, n$ and together form an open covering of F . Choose from this covering a finite one. If, however, there is a k , $1 \leq k \leq n$, such that not a single $U_{\alpha(k)}^{\xi} \subseteq G_k$ participates in the obtained covering, then adjoin to the covering an arbitrary element $U_{\alpha(k)}^{\xi}$ for which $F \cap U_{\alpha(k)}^{\xi} \neq \emptyset$. We obtain a covering

$$\{U_{\alpha(1)}^1, \dots, U_{\alpha(1)}^{p_1}, \dots, U_{\alpha(n)}^1, \dots, U_{\alpha(n)}^{p_n}\}$$

of the bicom pactum F , where

$$U_{\alpha(k)}^i \subseteq G_k, \quad k = 1, 2, \dots, n, \quad i = 1, 2, \dots, p_k. \quad (3)$$

Obviously,

$$F \in \langle U_{\alpha(1)}^1, \dots, U_{\alpha(1)}^{p_1}, \dots, U_{\alpha(n)}^1, \dots, U_{\alpha(n)}^{p_n} \rangle.$$

By virtue of (3) one obtains the inclusion

$$\langle U_{\alpha(1)}^1, \dots, U_{\alpha(1)}^{p_1}, \dots, U_{\alpha(n)}^1, \dots, U_{\alpha(n)}^{p_n} \rangle \subseteq \langle G_1, \dots, G_n \rangle.$$

The lemma is proved.

A system $\sigma = \{U_{\alpha}\}$ of open sets of the space X is called **dense** (Ponomarev) in this space if, for every open $G \subseteq X$, there exists $U_{\alpha} \in \sigma$ such that $U_{\alpha} \subseteq G$.

Lemma 2. The system $\Sigma = \{\langle U_{\alpha_1}, \dots, U_{\alpha_n} \rangle\}$, whose elements are nonempty finite collections of sets from the system $\sigma = \{U_{\alpha}\}$, dense in the space X , is dense in $\exp X$.

The least cardinal number that is the cardinality of an open base of the space X is called the **weight** of the space X and is denoted by wX .

Theorem 1. The weight of the space $\exp X$ coincides with the weight of X , if X is a bicom pactum and $wX \geq \aleph_0$.

The least cardinal number that is the cardinality of a system of open sets dense in X is called the π -**weight** of the space X and is denoted by $w_\pi X$ (Ponomarev).

Theorem 2. For an arbitrary topological space X ,

$$w_\pi X = w_\pi(\exp X), \quad \text{if } w_\pi X \geq \aleph_0.$$

Let $\mathfrak{B}_x = \{O_\alpha x\}$ be some fundamental system of neighborhoods of a point $x \in X$. The cardinal number $\inf_\alpha(wO_\alpha x)$ is called the **local weight of the point x in the space X** .

The space X is called **homogeneous with respect to local weight** if $w(x, X)$ is constant on X ⁽⁶⁾.

Theorem 3. The bicomcompactum $\exp X$ is homogeneous with respect to local weight and $w(F, \exp X) = \tau$ for every point $F \in \exp X$, if X is homogeneous with respect to local weight and $w(x, X) = wX = \tau \geq \aleph_0$.

Proof. A bicomcompactum X is homogeneous with respect to local weight and $w(x, X) = wX = \tau$ if and only if for every point $x \in X$ and any of its neighborhoods Ox there exists a bicomcompactum B_x such that $x \in B_x \subseteq Ox$, $wB_x = \tau$.

Take a point $F \in \exp X$ and any of its neighborhoods $\langle U_1, \dots, U_n \rangle$ from the base of $\exp X$. Since $\exp X$ is a bicomcompactum, there exists a smaller neighborhood $\langle V_1, \dots, V_s \rangle$ of the point F , for which

$$F \in \langle V_1, \dots, V_s \rangle \subseteq \langle \overline{V}_1, \dots, \overline{V}_s \rangle = \langle \overline{V}_1, \dots, \overline{V}_s \rangle \subseteq \langle U_1, \dots, U_n \rangle.$$

In each V_i , $i = 1, 2, \dots, s$, there is contained a bicomcompactum B_i of weight τ ; hence $w\overline{V}_i = \tau$. To obtain a base $\mathfrak{B}$ of the bicomcompactum $\langle \overline{V}_1, \dots, \overline{V}_s \rangle$, one must in each $\exp \overline{V}_i$ choose a base $\mathfrak{B}_i$ from sets of the form $\langle U_{\alpha_1}^i, \dots, U_{\alpha_n}^i \rangle$, $i = 1, 2, \dots, s$, where $\mathfrak{b}_i = \{U_\alpha^i\}$ is a base of $\overline{V}_i$, and unite them over $i = 1, 2, \dots, s$ into common brackets:

$$\langle U_{\alpha_1}^1, \dots, U_{\alpha_{p(1)}}^1, \dots, U_{\omega_1}^s, \dots, U_{\omega_{p(s)}}^s \rangle \in \mathfrak{B}.$$

If $|\mathfrak{B}_i| = \tau^*$, $i = 1, 2, \dots, s$, then $|\mathfrak{B}| = \tau$, and therefore

$$w\langle \overline{V}_1, \dots, \overline{V}_s \rangle \leq \tau.$$

Since all V_i are distinct, among them there will be some V_i , and in it a bicomcompactum B_i , such that $B_i \cap V_j = \emptyset$ for $i \neq j$, $wB_i = \tau$. The bicomcompactum $\exp B_i$ is topologically contained in $\langle \overline{V}_1, \dots, \overline{V}_s \rangle$. Since

$$wB_i = w(\exp B_i) = \tau,$$

it follows that

$$w\langle \overline{V}_1, \dots, \overline{V}_s \rangle = \tau.$$

The theorem is proved.

The **neighborhood character** $\chi(F, X)$ of a set F in the space X is the minimum of the cardinalities of fundamental systems of neighborhoods of F in X ⁽⁷⁾. The space X is called **homogeneous with respect to neighborhood character** if $\chi(x, X)$ is constant on X ⁽⁶⁾. The **pseudocharacter** $\psi(F, X)$ of a set F in X is the least number $\mathfrak{m}$ such that

$$F = \bigcap_{\alpha \in A} O_\alpha,$$

where all O_α are open and $|A| = \mathfrak{m}$ ⁽⁷⁾.

It is known that $\psi(F, X) = \chi(F, X)$ if F is closed and X is a bicom pactum.

Theorem 4. For an arbitrary topological space X and a closed $F \subseteq X$,

$$\chi(F, X) \leq \tau, \quad \text{if } \chi(F, \exp X) \leq \tau.$$

In ⁽⁷⁾ the following problem is formulated: does there exist a bicom pactum with the first axiom of countability, having cardinality $> \mathfrak{c}$, where $\mathfrak{c}$ is the cardinality of the continuum?

Theorem 5. Any bicom pactum $Y = \exp X$ satisfying the first axiom of countability has cardinality $\leq \mathfrak{c}$.

* The symbol $|A|$ denotes the cardinality of the set A .

Proof. Assuming the contrary, by means of Theorem 4 we conclude that the bicom pactum X is perfectly normal*. In that case $|X| \leq \mathfrak{c}$ ⁽⁷⁾. Since in X there are as many closed sets as open sets, and every open set, being of type F_σ , is the sum of no more than a countable number of closed sets, it follows that $|\exp X| \leq \mathfrak{c}$. The contradiction obtained proves the theorem.

A **caliber** of a space X is a cardinal number $\mathfrak{m}$ such that every system of nonempty open sets of the space X of cardinality $\mathfrak{m}$ has an equicardinal subsystem that has a nonempty intersection (Shanin).

It is known that every regular cardinal number $\mathfrak{m} > \aleph_0$ is a caliber of any dyadic bicom pactum ⁽⁶⁾.

Theorem 6. A regular cardinal number $\mathfrak{m} > \aleph_0$ is a caliber of the space $\exp X$ if and only if $\mathfrak{m}$ is a caliber of X .

Proof. We shall carry it out for a system of sets of an open base of the space $\exp X$. Let

$$\Sigma = \{\langle U_{\alpha_1}, \dots, U_{\alpha_s} \rangle\}$$

be a system of nonempty open sets in $\exp X$ of regular cardinality $\mathfrak{m} > \aleph_0$. It is assumed that the order of the elements in the brackets is fixed. The **length** of a set

$$\langle U_{\alpha_1}, \dots, U_{\alpha_s} \rangle \in \Sigma$$

is the number $s > 0$. By virtue of the regularity of the number $\mathfrak{m}$, there is a number $n > 0$ such that the system Σ contains an equicardinal subsystem $\Sigma_n^{(0)}$, all elements of which have the same length n . From each

$$\langle U_{\alpha_1}, \dots, U_{\alpha_n} \rangle \in \Sigma_n^{(0)}$$

the first element U_{α_1} is selected, and the system

$$\sigma_1 = \{U_{\alpha_1}\}$$

of the selected sets is considered in the space X . Since $\mathfrak{m}$ is a caliber of X , there exists

$$\sigma'_1 \subseteq \sigma_1, \quad |\sigma'_1| = |\sigma_1| = \mathfrak{m},$$

having a nonempty intersection in X . In $\exp X$ we consider the system

$$\Sigma_n^{(1)} \subseteq \Sigma_n^{(0)},$$

for whose elements $U_{\alpha_1} \in \sigma'_1$ in X . With the system $\Sigma_n^{(1)}$ the same is done as with $\Sigma_n^{(0)}$, but with respect to the second elements U_{α_2} . As a result one obtains a system

$$\Sigma_n^{(2)} \subseteq \Sigma_n^{(1)}.$$

Continuing the process described, at the n -th step we obtain a system

$$\Sigma_n^{(n)} \subseteq \Sigma_n^{(n-1)},$$

all n -th elements of whose sets belong to the system

$$\sigma'_n = \{U_{\alpha_n}\},$$

which has a nonempty intersection in X . Thus:

$$\Sigma \supseteq \Sigma_n^{(0)} \supseteq \Sigma_n^{(1)} \supseteq \dots \supseteq \Sigma_n^{(n)}; \quad (4)$$

$$|\Sigma_n^{(k)}| = |\Sigma| = \mathfrak{m}, \quad k = 0, 1, \dots, n; \quad (5)$$

$$\Sigma_n^{(n)} = \{\langle U_{\alpha_1}, \dots, U_{\alpha_n} \rangle\}, \quad U_{\alpha_k} \in \sigma'_k, \quad k = 1, 2, \dots, n. \quad (6)$$

In the space X , for each $k = 1, 2, \dots, n$, choose a point

$$x_k \in \bigcap_{\sigma'_k} U_{\alpha_k}.$$

Then

$$F = \{x_1, \dots, x_n\} \in \exp X,$$

$$F \in \bigcap_{\Sigma_n^{(n)}} \langle U_{\alpha_1}, \dots, U_{\alpha_n} \rangle.$$

Hence, and from (4), (5), the necessity of the condition follows. For the proof of sufficiency only the formula

$$\bigcap_{\alpha} \langle U_{\alpha} \rangle = \left\langle \bigcap_{\alpha} U_{\alpha} \right\rangle$$

is needed (2).

A topological space X satisfies the **Suslin condition** if every system of disjoint open sets from X is at most countable (see (6)).

Corollary 1. *The spaces X and $\exp X$ simultaneously satisfy the Suslin condition if $\aleph_1$ is a caliber of one of them.*

Corollary 2. *The space X satisfies the Suslin condition if the space $\exp X$ satisfies it.*

* A space X is called **perfectly normal** if it is normal and each of its closed sets is of type G_{δ} .

A topological space X satisfies the **Knaster condition** if every uncountable system of nonempty open subsets of X contains an uncountable subsystem every pair of whose sets has nonempty intersection (see (9)).

Theorem 7. *The topological space $\exp X$ satisfies the Knaster condition if and only if X satisfies this condition.*

The **density** sX of a space X is the least cardinality of everywhere dense subsets of the space X .

Theorem 8. *The densities of the spaces X and $\exp X$ coincide.*

Denote by T_{τ} the discrete T_1 -space of cardinality $\tau \geq \aleph_0$, and by $\mathfrak{b}_0 T_{\tau}$ its one-point compactification with vertex $\mathfrak{b}_0$ (the Alexandrov space) (see (1)).

Theorem 9. *The bicompactum $\exp \mathfrak{b}_0 T_{\tau}$ is a compactification of $\mathfrak{b} T_{\tau}$ with remainder $\mathfrak{b} T_{\tau} \setminus T_{\tau}$, homeomorphic to D^{τ} for every $\tau \geq \aleph_0$.*

Proof. Denote

$$X = \mathcal{E}(F \in \exp \mathfrak{b}_0 T_{\tau} : \mathfrak{b}_0 \notin F), \quad Y = \mathcal{E}(F \in \exp \mathfrak{b}_0 T_{\tau} : \mathfrak{b}_0 \in F).$$

Obviously, $X \cup Y = \exp \mathfrak{b}_0 T_{\tau}$, $X \cap Y = \emptyset$, X is open, and Y is closed in $\exp \mathfrak{b}_0 T_{\tau}$. It is not hard to see that X is homeomorphic to T_{τ} . Renumber the points $t \in T_{\tau}$:

$$T_{\tau} = \{t_{\alpha}\}_{\alpha \in A} | A| = \tau$$

and represent

$$D = \prod_{\alpha \in A} D_{\alpha}^{(0,1)}.$$

The mapping $f : Y \rightarrow D^{\tau}$ consists in assigning to each point $F \in Y$ the point $x \in D^{\tau}$ with coordinates

$$x_{\alpha} = 1, \quad \text{if } t_{\alpha} \in F \subseteq \mathfrak{b}_0 T_{\tau}, \quad x_{\alpha} = 0, \quad \text{if } t_{\alpha} \notin F.$$

It is easy to see that f is a homeomorphism onto all of D^τ .

Let $\aleph_0 \leq \mathfrak{m} < \tau$. The $\Sigma\mathfrak{m}$ -**product** is the set

$$\Sigma D^\tau \subset D^\tau$$

consisting of those points $x = \{x_\alpha\} \in D^\tau$ for which $x_\alpha \neq 0$ for at most a set of indices of cardinality $\leq \mathfrak{m}$ (6). Denote

$$\exp_{\mathfrak{m}} X = \mathcal{E}(F \in \exp X : |F| \leq \mathfrak{m})$$

for any cardinal number $\mathfrak{m} \geq \aleph_0$.

Corollary. In $\exp \mathfrak{b}_0 T_\tau$,

$$(\exp \mathfrak{b}_0 T_\tau \setminus T_\tau) \cap \exp_{\mathfrak{m}} \mathfrak{b}_0 T_\tau \cong \Sigma_{\mathfrak{m}} D^\tau.$$

The dimension of the space Y sometimes allows one to judge whether Y is representable in the form $\exp X$.

Theorem 10. The compactum $\exp X$ is infinite-dimensional if $\dim X \geq 1$.

Proof. The mapping

$$f_n : X^n \rightarrow \exp_n X$$

is (4) open-and-closed and $n!$ -to-one. Therefore

$$\dim X^n = \dim(\exp_n X).$$

Using results from (8), it is easy to obtain the inequality, for any $n < \aleph_0$:

$$\dim(\exp_n X) = \dim X^n \geq n, \quad \text{if } \dim X \geq 1.$$

Since $\exp_n X$ is closed in $\exp X$ for every $n = 1, 2, \dots$ and $\dim(\exp_n X) \geq n$, the space $\exp X$ is an infinite-dimensional compactum.

Corollary. The space

$$\exp_{\aleph_0} X = \mathcal{E}(F \in \exp X : |F| < \aleph_0)$$

of all finite subsets of the compactum X is weakly infinite-dimensional.

It remains an open question: will $\exp X$ be a strongly infinite-dimensional compactum if $\dim X \geq 1$?

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$$* D^\tau$$

is the generalized Cantor set, i.e. the Tikhonov product of τ discrete two-point spaces, $\tau \geq \aleph_0$.

Note: Figure translations are in progress. See original paper for figures.

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