

ON THE EXISTENCE OF MOMENTS OF THE NUMBER OF LEVEL CROSSINGS BY A GAUSSIAN STATIONARY PROCESS

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Abstract

Full Text

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MATHEMATICS

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ON THE EXISTENCE OF MOMENTS OF THE NUMBER OF LEVEL CROSSINGS BY A GAUSSIAN STATIONARY PROCESS

(Presented by Academician A. N. Kolmogorov on 5 I 1968)

It is known ^(1,2) that the m -th factorial moment of the number of crossings from below upward of the level a by a Gaussian process without linear relations ξ_t on the interval Δ is equal to

$$d_m(\Delta, a) = \int_{\{t_i \in \Delta, t_i \neq t_j, i, j=1, \dots, m\}} M \left\{ \prod_{i=1}^m \dot{\xi}_{t_i}^+ \mid \xi_{t_j} = a, j = 1, \dots, m \right\} \times P_{t_1 \dots t_m}(a, \dots, a) dt_1 \dots dt_m, \quad (1)$$

where $P_{t_1 \dots t_m}(x_1, \dots, x_m)$ is the density of the joint distribution of the quantities $\xi_{t_1}, \dots, \xi_{t_m}$ with covariance matrix $R_{11}(t_1, \dots, t_m)$,

$$\dot{\xi}_{t_i} = d\xi_{t_i}/dt_i, \quad \dot{\xi}_{t_i}^+ = \frac{1}{2}(\dot{\xi}_{t_i} + |\dot{\xi}_{t_i}|).$$

Below we study the question of conditions for the existence of the factorial moments specified by the integrals (1). Denote

$$I(l) = \int_{\{t_i \in \Delta, t_i \neq t_j, i, j=1, \dots, m\}} \frac{\mu_{i_1} \dots \mu_{i_l} \sigma_{i_{l+1}} \dots \sigma_{i_m}}{\sqrt{|R_{11}(t_1, \dots, t_m)|}} dt_1, \dots, dt_m,$$

where

$$\mu_i = M\{\dot{\xi}_{t_i} \mid \xi_{t_j} = a, j = 1, \dots, m\}, \quad \sigma_i^2 = D\{\dot{\xi}_{t_i} \mid \xi_{t_j} = a, j = 1, \dots, m\},$$

$(i_1, \dots, i_m)$ is a permutation of the indices $(1, \dots, m)$.

Theorem 1. *For the existence of the m -th moment of the number of crossings of the level a by a Gaussian process it is sufficient that the integrals $I(l)$ ($l = 0, \dots, m$) exist, and necessary that the integral $I(0)$ exist.*

Sufficiency was proved by Yu. K. Belyaev ⁽¹⁾, Lemma 4. Necessity follows from two lemmas.

Lemma 1. *For any vector a and nondegenerate matrix A ,*

$$a' A^{-1} a = \min_x (x' A x - 2a' x). \quad (2)$$

Lemma 2. *There exist $\varepsilon > 0$ and $0 < \alpha < 1$ such that, if $\max_{i,j} |t_i - t_j| < \varepsilon$, then*

$$M \left\{ \prod_{i=1}^m \dot{\xi}_{t_i}^+ \mid \xi_{t_j} = a, j = 1, \dots, m \right\} \geq \alpha \sigma_1 \cdots \sigma_m.$$

Theorem 2. *Suppose:*

1. *The k -th mean-square derivative of the stationary Gaussian process exists.*
2. *In some interval $(0, \delta)$ there exist and are continuous $\rho^{(2k+1)}(t)$ and $\rho^{(2k+2)}(t)$, respectively the $(2k+1)$ -st and $(2k+2)$ -nd derivatives of the correlation function of ξ_t .*
3. *There exist limits*

$$\lim_{t \downarrow 0} \frac{\rho^{(2k)}(0) - \rho^{(2k)}(t)}{t} = \rho_+^{(2k+1)}(0) \neq 0,$$

$$\lim_{t \downarrow 0} \frac{\rho_+^{(2k+1)}(0) - \rho^{(2k+1)}(t)}{t} = \rho_+^{(2k+2)}(0).$$

Under these conditions $\alpha_{(m)}(\Delta, a) < \infty$ if and only if $m \leq k^2 + 2k$.

Proof. We find the asymptotics, as $t_i - t_j \rightarrow 0$, of the functions

$$\frac{\mu_{i_1} \cdots \mu_{i_l} \sigma_{i_{l+1}} \cdots \sigma_{i_m}}{\sqrt{|R_{11}(t_1, \dots, t_m)|}}. \quad (3)$$

We shall carry out the estimate when all $t_i - t_j \rightarrow 0$. In the remaining cases the estimate is analogous. At the same time, for brevity, by $\lim$ we mean passage to the limit as $\max |t_i - t_j| \rightarrow 0$.

Consider the quantities

$$\zeta_{t_i t_{i+1}}^{(k)} = \frac{\xi_{t_{i+1}}^{(k)} - \xi_{t_i}^{(k)}}{\sqrt{|t_{i+1} - t_i|}}.$$

It is easy to see that

$$\lim M \zeta_{t_i t_{i+1}}^{(k)} \zeta_{t_i t_{i+1}}^{(k)} = \begin{cases} 0, & \text{if } i \neq l, \\ 2\rho_+^{(2k+1)}(0), & \text{if } i = l. \end{cases}$$

Denote, following (1), by $L_{t_1 \dots t_{k+1}}[\xi_t] = [\xi_{t_1}, \dots, \xi_{t_{k+1}}]/k!$, where $[\xi_{t_1}, \dots, \xi_{t_k}]$ is the k -th divided difference of ξ_t .

The quantities

$$\zeta_{t_1 \dots t_{k+2}} = \frac{L_{t_1 \dots t_{k+1}}[\xi_t] - L_{t_2 \dots t_{k+2}}[\xi_t]}{\sqrt{|t_1 - t_{k+2}|}}$$

possess the same properties as the $\zeta_{t_i t_{i+1}}^{(k)}$ introduced above.

It is known that

$$\mu_i = \frac{|R_{\mu_i}|}{|R_{11}|} = \left| \begin{array}{c|c} R_{11} & \frac{\partial \rho(t_i - t_j)}{\partial t_i} \\ \dots & \dots \\ a & \dots & a \\ \hline & 0 \end{array} \right| |R_{11}(t_1, \dots, t_m)|^{-1},$$

$$\sigma_i^2 = \frac{|R_{\sigma_i}|}{|R_{11}|} = \left| \begin{array}{c|c} R_{11} & \frac{\partial \rho(t_i - t_j)}{\partial t_i} \\ \dots & \dots \\ \dots & \frac{\partial \rho(t_i - t_j)}{\partial t_i} & \dots \\ \hline & \frac{\partial^2 \rho(0)}{\partial t^2} \end{array} \right| |R_{11}(t_1, \dots, t_m)|^{-1}.$$

We perform the following transformations on $|R_{11}|$. At the first step, from each i -th column we subtract the $(i-1)$ -st and divide by $t_i - t_{i-1}$, and from each i -th row we subtract the $(i-1)$ -st and divide by $t_i - t_{i-1}$.

At the second step we do the same as at the first, without touching the first row and column, and dividing by $t_i - t_{i-2}$, and so on up to the k -th step (see (1)). At the $(k+1)$ -st step, from the i -th column ($i = m, \dots, k+2$) we subtract the $(i-1)$ -st and divide by $\sqrt{|t_i - t_{i-k-1}|}$; we do the same with the rows and pass to the limit.

Having carried out analogous transformations on R_{σ_i} and R_{μ_i} , we obtain asymptotic formulas.

$$\frac{1}{2^{m-k-1}} \lim \frac{|R_{11}(t_1, \dots, t_m)|}{\prod_{l=1, \dots, k} \prod_{i=l+1, \dots, m} (t_i - t_{i-l})^2 \prod_{i=k+2, \dots, m} |t_i - t_{i-k-1}|} = |M\eta\eta'| [\rho_+^{(2k+1)}(0)]^{m-k-1},$$

$$\lim \frac{|R_{\sigma_i}(t_1, \dots, t_m)|}{|R_{11}(t_1, \dots, t_m)| \prod_{l=1, \dots, k-1} (t_i - t_{i-l}) |t_i - t_{i-k}|} = 2\rho_+^{(2k+1)}(0),$$

$$\lim \frac{|R_{\mu_i}(t_1, \dots, t_m)|}{|R_{11}(t_1, \dots, t_m)| \prod_{l=1, \dots, k-1} (t_i - t_{i-l}) \sqrt{|t_i - t_{i-k}|}} = k\rho_+^{(2k+1)}(0),$$

where $\eta = (\xi_{t_0}, \dots, \xi_{t_0}^{(k)})$.

On the basis of these formulas we conclude that, for $m = k^2 + 2k$, the singularities of the function (3) are integrable, whereas already for $m = k^2 + 2k + 1$ $I(0)$ does not exist.

Corollary. *For Gaussian stationary processes with rational spectral density that have exactly k derivatives in the mean-square sense, the last finite moment of the number of level crossings has order $k^2 + 2k$.*

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