

STATISTICAL DYNAMICS OF A TURBULENT INCOMPRESSIBLE FLUID

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1968

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Abstract

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UDC 532.5.045

HYDROMECHANICS

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STATISTICAL DYNAMICS OF A TURBULENT INCOMPRESSIBLE FLUID

(Presented by Academician M. A. Leontovich on 11 IV 1968)

I. Basic equations. Elimination of pressure. The Navier–Stokes equations for describing the motion of a viscous incompressible fluid have the form:

$$\partial v_\alpha / \partial t + v_\beta \partial v_\alpha / \partial x_\beta + \partial p / \partial x_\alpha = \nu \Delta v_\alpha, \quad \partial v_\beta / \partial x_\beta = 0. \quad (1)$$

From these two equations the pressure can be eliminated in the usual way:

$$p(\mathbf{x}) = -\frac{1}{4\pi} \int \frac{\partial v'_\alpha}{\partial x'_\beta} \frac{\partial v'_\beta}{\partial x'_\alpha} \frac{d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} + \psi(\mathbf{x}). \quad (2)$$

The function $\psi(\mathbf{x})$ is harmonic,

$$\Delta \psi(\mathbf{x}) = 0. \quad (3)$$

Although in what follows we shall regard the velocity field as random, we shall show that the function ψ is not a random variable. We shall denote averaging over the ensemble by an overbar and introduce a new random function

$$\psi_1(\mathbf{x}) = \psi(\mathbf{x}) - \overline{\psi(\mathbf{x})}$$

and its two-point second moment

$$\overline{\psi_1(\mathbf{x})\psi_1(\mathbf{x} + \mathbf{r})} = f(\mathbf{x}, \mathbf{r}). \quad (4)$$

For fixed $\mathbf{x}$, the function $f(\mathbf{x}, \mathbf{r})$ is harmonic. In addition, it satisfies the following physically evident requirements: it must have no singularities and must tend to zero for large $\mathbf{r}$. A harmonic function possessing such properties is equal to zero. Passing in formula (4) to the limit $\mathbf{r} \rightarrow 0$, we obtain

$$\overline{\psi_1^2(\mathbf{x})} = 0,$$

which means that the random function ψ coincides exactly with its mean value:

$$\psi(\mathbf{x}) \equiv \overline{\psi(\mathbf{x})}.$$

Thus, in our equations we may regard $\psi(\mathbf{x})$ as not being a random variable.

After integration by parts in formula (2), the equations acquire the form

$$\begin{aligned} \frac{\partial v_\alpha}{\partial t} + v_\beta \frac{\partial v_\alpha}{\partial x_\beta} - \frac{1}{4\pi} \int v_\beta(\mathbf{x}') v_\gamma(\mathbf{x}') T_{\alpha\beta\gamma}(\mathbf{x} - \mathbf{x}') d\mathbf{x}' &= -\frac{\partial \bar{\psi}}{\partial x_\alpha} + \nu \Delta v_\alpha, \\ \frac{\partial v_\beta}{\partial x_\beta} &= 0, \end{aligned} \quad (5)$$

where

$$T_{\alpha\beta\gamma}(\mathbf{x} - \mathbf{x}') = \frac{\partial^3}{\partial x_\alpha \partial x_\beta \partial x_\gamma} \frac{1}{|\mathbf{x} - \mathbf{x}'|}.$$

II. Chain of equations for distribution functions. Introduce distribution functions statistically describing the behavior

fluids such that the probability that at the points $\mathbf{x}_1, \dots, \mathbf{x}_n$ at the times $t_1, \dots, t_n$, respectively, the fluid velocities will lie in the intervals $d\mathbf{v}_1, \dots, d\mathbf{v}_n$ is

$$F_n(\mathbf{v}_1, \dots, \mathbf{v}_n; \mathbf{x}_1, \dots, \mathbf{x}_n; t_1, \dots, t_n) d\mathbf{v}_1 \dots d\mathbf{v}_n. \quad (6)$$

It is clear that the functions F_n give a complete statistical description of the turbulent motion of a fluid. Knowing F_n , one can find any mean characteristics, for example the mean dissipated energy

$$\begin{aligned} \varepsilon(\mathbf{x}_1, t_1) &= \nu \frac{\partial v_\alpha(\mathbf{x}_1, t_1)}{\partial x_{1\beta}} \frac{\partial v_\alpha(\mathbf{x}_1, t_1)}{\partial x_{1\beta}} = \\ &= \nu \int \frac{\partial^2 F_2(\mathbf{v}_1, \mathbf{v}_2; \mathbf{x}_1, \mathbf{x}_2; t_1, t_2)}{\partial x_{1\beta} \partial x_{2\beta}} v_{1\alpha} v_{2\alpha} \delta(\mathbf{x}_1 - \mathbf{x}_2) \delta(t_1 - t_2) d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{x}_2 dt_2. \end{aligned}$$

Let us derive the chain of equations satisfied by the functions F_n . To this end consider an arbitrary function $\varphi(\mathbf{v}_1(\mathbf{x}_1, t'), \dots, \mathbf{v}_n(\mathbf{x}_n, t_n))$ of the velocities at the points $\mathbf{x}_1, t_1, \dots, \mathbf{x}_n, t_n$ of space-time. It is clear that

$$\frac{\partial \varphi}{\partial t_s} = \frac{\partial \varphi}{\partial v_{s\alpha}} \frac{\partial v_{s\alpha}(\mathbf{x}_s, t_s)}{\partial t_s},$$

or, using equation (5), we can write

$$\begin{aligned} \frac{\partial \varphi}{\partial t_s} = \frac{\partial \varphi}{\partial x_{s\alpha}} \left[-v_{s\beta}(\mathbf{x}_s, t_s) \frac{\partial v_{s\alpha}(\mathbf{x}_s, t_s)}{\partial x_{s\beta}} + \right. \\ \left. + \frac{1}{4\pi} \int v'_\beta(\mathbf{x}', t_s) v'_\gamma(\mathbf{x}', t_s) T_{\alpha\beta\gamma}(\mathbf{x}_s - \mathbf{x}') d\mathbf{x}' + \nu \Delta v_{s\alpha}(\mathbf{x}_s, t_s) \right]. \end{aligned} \quad (7)$$

Take the mean of relation (7). We have

$$\begin{aligned} \overline{\frac{\partial \varphi}{\partial t_s}} &= \int \frac{\partial F_n}{\partial t_s} \varphi d\mathbf{v}_1 \dots d\mathbf{v}_n, \\ \overline{\frac{\partial \varphi}{\partial v_{s\alpha}} v_{s\beta} \frac{\partial v_{s\alpha}}{\partial x_{s\beta}}} &= \int \frac{\partial F_n}{\partial x_{s\beta}} v_{s\beta} \varphi(\mathbf{v}_1, \dots, \mathbf{v}_n) d\mathbf{v}_1 \dots d\mathbf{v}_n. \end{aligned}$$

The other terms in relation (7) are transformed analogously. Using the fact that φ is an arbitrary function, we obtain the chain of equations for the functions F_n

$$\begin{aligned} \frac{\partial F_n}{\partial t_s} &= -v_{s\beta} \frac{\partial F_n}{\partial x_{s\beta}} - \frac{1}{4\pi} \int \frac{\partial F_{n+1}}{\partial v_{s\alpha}} v_{n+1,\beta} v_{n+1,\gamma} T_{\alpha\beta\gamma}(\mathbf{x}_s - \mathbf{x}_{n+1}) \times \\ &\times \delta(t_{n+1} - t_s) d\mathbf{v}_{n+1} dt_{n+1} d\mathbf{x}_{n+1} - \frac{\partial F_n}{\partial v_{s\alpha}} \frac{\partial \bar{\psi}(\mathbf{x}_s, t_s)}{\partial x_{s\alpha}} - \\ &- \nu \int \delta(\mathbf{x}_s - \mathbf{x}_{n+1}) \delta(t_s - t_{n+1}) d\mathbf{x}_{n+1} \Delta_{n+1} \int \frac{\partial F_{n+1}}{\partial v_{s\alpha}} v_{n+1,\alpha} d\mathbf{v}_{n+1} dt_{n+1} * . \end{aligned} \quad (8)$$

If one restricts oneself only to simultaneous functions, then in the right-hand sides of equations (8) one must pass to the limit $t_1 = t_2 = \dots = t$, take into account that

$$\lim_{t_1 \rightarrow t, t_2 \rightarrow t, \dots, t_n \rightarrow t} \left[\sum_{s=1}^n \frac{\partial F_n}{\partial t_s} \right] = \frac{\partial F_n(\mathbf{v}_1, \dots, \mathbf{v}_n; \mathbf{x}_1, \dots, \mathbf{x}_n; t)}{\partial t}$$

and obtain for the simultaneous correlation functions the chain of equa-

* Equations (8), as we learned while preparing the article for press, were obtained by another method for the

$$\begin{aligned} \frac{\partial F_n}{\partial t} = & - \sum_{k=1}^n v_{k\beta} \frac{\partial F_n}{\partial x_{k\beta}} - \frac{1}{4\pi} \sum_{k=1}^n \int \frac{\partial F_{n+1}}{\partial v_{k\alpha}} v_{n+1,\beta} v_{n+1,\gamma} T_{\alpha\beta\gamma}(\mathbf{x}_k - \mathbf{x}_{n+1}) d\mathbf{v}_{n+1} d\mathbf{x}_{n+1} - \\ & - \nu \sum_{k=1}^n \int \delta(\mathbf{x}_k - \mathbf{x}_{n+1}) d\mathbf{x}_{n+1} \Delta_{n+1} \int \frac{\partial F_{n+1}}{\partial v_{k\alpha}} v_{n+1,\alpha} d\mathbf{v}_{n+1}. \end{aligned} \quad (9)$$

III. Additional conditions

We present the conditions that the functions F_n must satisfy.

1. Normalization conditions:

$$\int F_{n+1} d\mathbf{v}_{n+1} = F_n, \quad \int F_n d\mathbf{v}_1 \dots d\mathbf{v}_n = 1.$$

2. Continuity conditions:

$$\lim_{\mathbf{x}_{n+1} \rightarrow \mathbf{x}_n, t_{n+1} \rightarrow t_n} F_{n+1} = F_n \delta(\mathbf{v}_{n+1} - \mathbf{v}_n).$$

3. Symmetry conditions:

$$\begin{aligned} F_n(\dots, \mathbf{v}_s, \dots, \mathbf{v}_k, \dots; \dots, \mathbf{x}_s, \dots, \mathbf{x}_k, \dots; \dots, t_s, \dots, t_k, \dots) = \\ = F_n(\dots, \mathbf{v}_k, \dots, \mathbf{v}_s, \dots; \dots, \mathbf{x}_k, \dots, \mathbf{x}_s, \dots; \dots, t_k, \dots, t_s, \dots). \end{aligned}$$

4. Incompressibility condition:

$$\int \frac{\partial F_n}{\partial x_{k\alpha}} v_{k\alpha} d\mathbf{v}_k = 0.$$

5. Compatibility conditions:

$$\frac{\partial F_n}{\partial x_{k\alpha}} = - \frac{\partial}{\partial v_{k\beta}} \int \delta(\mathbf{x}_{n+1} - \mathbf{x}_k) \frac{\partial F_{n+1}}{\partial x_{n+1,\alpha}} v_{n+1,\beta} d\mathbf{v}_{n+1} d\mathbf{x}_{n+1} \delta(t_{n+1} - t_k) dt_{n+1},$$

$$\frac{\partial F_n}{\partial t_k} = - \frac{\partial}{\partial v_{k\beta}} \int \delta(\mathbf{x}_{n+1} - \mathbf{x}_k) \delta(t_{n+1} - t_k) \frac{\partial F_{n+1}}{\partial t_{n+1}} v_{n+1,\beta} d\mathbf{v}_{n+1} d\mathbf{x}_{n+1} dt_{n+1}.$$

Only when all the conditions are fulfilled can the functions F_n be regarded as functions with independent variables.

Received
28 I 1968

CITED LITERATURE

1. A. S. Monin, PMM, **31**, no. 6 (1967).

Note: Figure translations are in progress. See original paper for figures.

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