

ON THE THEORY OF CANONICAL DIFFERENTIAL OPERATORS IN HILBERT SPACE

V. M. Adamyan

1968

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196801.24701>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.43:517.925.2

MATHEMATICS

V. M. ADAMYAN

ON THE THEORY OF CANONICAL DIFFERENTIAL OPERATORS IN HILBERT SPACE

(Presented by Academician V. M. Glushkov on 6 III 1967)

1. Let $\mathfrak{H}$ be a separable Hilbert space, and let $\mathfrak{B}$ be the ring of continuous operators in $\mathfrak{H}$. For any interval E of the real axis and any subspace $\mathfrak{H}' \subset \mathfrak{H}$, by $L_2(E, \mathfrak{H}')$ we shall denote the space of measurable functions $f(x)$, $x \in E$, with values in $\mathfrak{H}'$, satisfying the condition

$$\int_E \|f(x)\|_{\mathfrak{H}}^2 dx < \infty.$$

By $L(E, \mathfrak{B})$ we shall denote the space of functions with values in $\mathfrak{B}$ that are Bochner-integrable on E .

Choose in $\mathfrak{B}$ some operator J with the properties: $J^* = -J$, $J^2 = -I$, where I is the identity operator. For J , evidently, there is the spectral decomposition $J = iP_+ - iP_-$, where $P_{\pm}$ are orthogonal projectors, $P_{\pm} = \frac{1}{2}(J \pm iI)$, which decompose $\mathfrak{H}$ into two subspaces $\mathfrak{H}_+$ and $\mathfrak{H}_-$, $\mathfrak{H}_{\pm} = P_{\pm}\mathfrak{H}$. We assume that $\dim \mathfrak{H}_+ = \dim \mathfrak{H}_-$. By virtue of this assumption one can construct an isometric operator Z mapping $\mathfrak{H}_-$ onto $\mathfrak{H}_+$, and form the subspace $\mathfrak{H}_0 = \{h : h = e + Ze, e \in \mathfrak{H}_-\}$. From the definition of $\mathfrak{H}_0$ it follows that $(Jf, g) = 0$ for all $f, g \in \mathfrak{H}_0$ and that the subspace $\mathfrak{H}_0$ admits no extensions in $\mathfrak{H}$ with preservation of this property. In what follows the operators J and Z are regarded as fixed.

The purpose of the present note is to study the self-adjoint differential operator A in $L_2((0, \infty); \mathfrak{H})$, formally defined by the expression

$$(Af)(x) = -J df(x)/dx + V(x)f(x) \quad (1)$$

and by the boundary condition

$$f(0) \in \mathfrak{H}_0, \quad (2)$$

where $V(x)$ is a function from $L(0, \infty; \mathfrak{B})$ with self-adjoint values. We also impose on $V(x)$ the additional condition

$$JV(x) = -V(x)J, \quad (3)$$

which in fact does not impair the generality of the consideration, since one can always achieve its fulfillment by passing from A to an isomorphic operator G^*AG , where $G = G(x)$ is a function whose values are unitary operators in $\mathfrak{R}$, defined by the equalities

$$-J dG/dx = -\{P_+ V(x)P_+ + P_- V(x)P_-\}G, \quad G(0) = I.$$

Differential operators of the form (1) under the condition $\dim \mathfrak{R} < \infty$, i.e., in those cases to which the radial equations of quantum theory for relativistic particles and, in particular, the radial Dirac equation lead, have already been considered in works ⁽¹⁻³⁾, where, under the assumption of local summability of the function $V(x)$, the direct problem of scattering theory was studied and a solution was found to the inverse problems of reconstructing $V(x)$ from the spectrum and from scattering data. (Some particulars and details concerning ⁽²⁾ can be found in the article ⁽⁴⁾.) Inverse problems for the radial Dirac equation were also studied in recent works ^(5,6).

Below, without assuming that $\dim \mathfrak{R} < \infty$, but requiring that $V(x) \in L(0, \infty; \mathfrak{B})$, we give for the operator A an analogue of the eigenfunction expansion theorem and a solution of the direct problem of scattering theory, and, in the case when the function $V(x)$ is finite, we establish the applicability to the operator A of the variant of scattering theory proposed by P. Lax and R. Phillips ⁽⁷⁾. We also give some results concerning operators of the form (1) with a non-self-adjoint function $V(x)$.

2. Consider the differential equation

$$-J dF(x, \lambda)/dx + V(x)F(x, \lambda) - \lambda F(x, \lambda) = 0, \quad (4)$$

where λ is a real parameter. A function $F(x, \lambda)$ with values in $\mathfrak{B}$ will be called a solution of equation (4) if, for $F(x, \lambda)$, equality (4) holds in the norm of the space $\mathfrak{B}$.

Denote by $\mathcal{E}(x, \lambda)$ the solution of equation (4) satisfying the condition $\mathcal{E}(0, \lambda) = I$. Note that there exists only one such function $\mathcal{E}(x, \lambda)$, and every function $F(x, \lambda)$ satisfying (4) and the condition $F(0, \lambda) = C$, where $C \in \mathfrak{B}$, is obtained from the relation $F(x, \lambda) = \mathcal{E}(x, \lambda)C$.

For $\mathcal{E}(x, \lambda)$ the equalities

$$\mathcal{E}^*(x, \lambda)J\mathcal{E}(x, \lambda) = \mathcal{E}(x, \lambda)J\mathcal{E}^*(x, \lambda) = J \quad (5)$$

and the representation

$$\mathcal{E}(x, \lambda) = \exp(J\lambda x) + \int_0^x \exp[J\lambda(x-s)] \Gamma(x, s) ds, \quad (6)$$

hold, where $\Gamma(x, s)$ ($0 < s < x$, $0 < x < \infty$) belongs to the space of functions with values in $\mathfrak{B}$, measurable in the norm of $\mathfrak{B}$, and such that

$$\sup_x \int_0^x \|\Gamma(x, s)\| ds < \infty.$$

For the determination of $\Gamma(x, s)$ one uses the “integral” equation

$$\Gamma(x, s) = -JV(s) - J \int_s^x V(t) \Gamma(t, t-s) dt, \quad (7)$$

which has a unique solution in the indicated space of functions, provided $V(s) \in L(0, x; \mathfrak{B})$.

As $x \rightarrow \infty$, the sequence $\{\exp(-J\lambda x)\mathcal{E}(x, \lambda)\}$ converges uniformly in λ , and

$$G(\lambda) = \lim_{x \rightarrow \infty} \exp(-J\lambda x)\mathcal{E}(x, \lambda) = I + \int_0^\infty \exp(-J\lambda s) \Gamma(s) ds,$$

where $\Gamma(s)$ is the limit in $L(0, \infty; \mathfrak{B})$ of the convergent sequence of functions $\{\Gamma_x(s)\}$, and $\Gamma_x(s) = \Gamma(x, s)$ for $x > s$ and $\Gamma_x(s) = 0$ for $x < s$.

The estimates

$$\exp \left\{ - \int_0^\infty \|V(s)\| ds \right\} \cdot I \leq \{G^*(\lambda)G(\lambda)\}^{1/2} \leq \exp \left\{ \int_0^\infty \|V(s)\| ds \right\} \cdot I \quad (8)$$

are valid.

3. Let $\Phi(x, \lambda)$ ($\text{Im } \lambda = 0$) be the solution of equation (4) satisfying the condition $\Phi(0, \lambda) = P_0$, where P_0 is the orthogonal projector onto the subspace $\mathfrak{R}_0$, i.e. $\Phi(x, \lambda) = \mathcal{E}(x, \lambda)P_0$. By virtue of (8), the self-adjoint operator $P_0 G^*(\lambda)G(\lambda)P_0$ is continuously invertible in $\mathfrak{R}_0$, and for the corresponding inverse operator $\Delta(\lambda)$ the estimates

$$\exp \left\{ -2 \int_0^\infty \|V(s)\| ds \right\} P_0 \leq \Delta(\lambda) \leq \exp \left\{ 2 \int_0^\infty \|V(s)\| ds \right\} P_0. \quad (9)$$

hold.

Denote by $L_2^\Delta(-\infty, \infty; \mathfrak{N}_0)$ the space of measurable functions with values in $\mathfrak{N}_0$ in which the norm is given by the formula

$$\|f\|_\Delta = \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} (\Delta(\lambda)f(\lambda), f(\lambda)) d\lambda \right\}^{1/2}.$$

Inequality (9) shows that the spaces $L_2(-\infty, \infty; \mathfrak{N}_0)$ and $L_2^\Delta(-\infty, \infty; \mathfrak{N}_0)$ consist of the same functions.

Theorem 1. For any function $f(x) \in L_2(0, \infty; \mathfrak{N})$ the representation

$$f(x) = \text{l. i. m.}_{N \rightarrow \infty} \frac{1}{\pi} \int_{-N}^N \Phi(x, \lambda) \Delta(\lambda) \tilde{f}(\lambda) d\lambda, \quad (10)$$

is valid, where $\tilde{f}(\lambda) \in L_2^\Delta(-\infty, \infty; \mathfrak{N}_0)$ and is found by the inversion formula

$$\tilde{f}(\lambda) = \text{l. i. m.}_{R \rightarrow \infty} \int_0^R \Phi^*(x, \lambda) f(x) dx. \quad (11)$$

There is an analogue of Parseval's equality

$$\int_0^\infty \|f(x)\|^2 dx = \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda (\Delta(\lambda) \tilde{f}(\lambda), \tilde{f}(\lambda)). \quad (12)$$

The assertion of Theorem 1 follows from the relations and estimates given above if, in addition, one assumes that $V(x) = 0$ for $x > a > 0$ and $\sup_x \|V(x)\| < \infty$. In the general case it is enough to note that the set of functions $V(x)$ from $L(0, \infty; \mathfrak{B})$ with self-adjoint values for which Theorem 1 is true is closed in $L(0, \infty; \mathfrak{B})$.

Under the mapping (10) of the space $L_2^\Delta(-\infty, \infty; \mathfrak{N}_0)$ onto $L_2(0, \infty; \mathfrak{N})$, the operator of multiplication by the variable λ in $L_2^\Delta(-\infty, \infty; \mathfrak{N}_0)$ generates a self-adjoint operator A , defined on the set dense in $L_2(0, \infty; \mathfrak{N})$ which is the image of the set

$$\left\{ f(\lambda); \frac{1}{\pi} \int_{-\infty}^{\infty} \lambda^2 (\Delta(\lambda) f(\lambda), f(\lambda)) d\lambda \right\}.$$

By the self-adjoint operator A generated by the differential expression (1) and condition (2), one should understand precisely this operator. The operator A has a homogeneous Lebesgue spectrum of multiplicity $\dim \mathfrak{N}_0 = \dim \mathfrak{N}_\pm$.

4. Denote by A_0 the differential operator of the type under consideration corresponding to the case $V(x) \equiv 0$.

Theorem 2. There exist and are unitary wave operators

$$W_\pm(A, A_0) = s\text{-}\lim_{t \rightarrow \pm\infty} e^{iAt} e^{-iA_0 t}. \quad (13)$$

The action of the operators $W_{\pm}(A, A_0)$ is determined by the formulas

$$\begin{aligned} (W_{\pm}(A, A_0)f)(x) &= \text{l. i. m.}_{N \rightarrow \infty} \int_{-N}^N \Phi(x, \lambda) \Delta(\lambda) A_{\mp}(\lambda) \tilde{f}(\lambda) d\lambda, \\ \tilde{f}(\lambda) &= \text{l. i. m.}_{R \rightarrow \infty} P_0 \int_0^R \exp\{-J\lambda x\} f(x) dx, \end{aligned} \quad (14)$$

where

$$A_{\mp}(\lambda) = 2P_0 G^*(\lambda) P_{\mp} P_0.$$

From the formal properties of wave operators and Theorem 2 it follows that

Corollary. The operators A and A_0 are isomorphic, and

$$W_{\pm}(A, A_0) E_{\lambda}^0 W_{\pm}^*(A, A_0) = E_{\lambda},$$

where E_{λ} and E_{λ}^0 are the resolutions of the identity of the operators A and A_0 .

Using the wave operators, we construct the scattering operator

$$S = (A, A_0) = W_+^*(A, A_0) W_-(A, A_0).$$

On the basis of (14), for any function $f(x) \in L_2(0, \infty; \mathfrak{N})$ we have

$$\begin{aligned} (S(A, A_0)f)(x) &= \text{l. i. m.}_{N \rightarrow \infty} \frac{1}{\pi} P_0 \int_{-N}^N \exp(J\lambda x) A_-^{-1}(\lambda) A_+(\lambda) \tilde{f}(\lambda) d\lambda, \\ \tilde{f}(\lambda) &= \text{l. i. m.}_{R \rightarrow \infty} P_0 \int_0^R \exp(-J\lambda x) f(x) dx. \end{aligned} \quad (15)$$

It is seen from (15) that the scattering suboperator (the Geisenberg scattering matrix) is computed by the formula

$$S(\lambda) = A_-(\lambda) A_+^{-1}(\lambda). \quad (16)$$

Let us note that the functions $A_+(\lambda)$ and $A_-(\lambda)$ are boundary values of functions analytic respectively in the upper and lower half-planes and factorize the spectral density $\Delta(\lambda)$ in the form

$$\Delta(\lambda) = (A_{\pm}(\lambda) A_{\pm}^*(\lambda))^{-1}. \quad (17)$$

5. Suppose that $V(x) = 0$ for $x > a > 0$, and denote by $\mathfrak{D}_{\pm}$ the subspaces $L_2(0, \infty; \mathfrak{N}_{\pm})$ of the space $L_2(0, \infty; \mathfrak{N})$. With respect to the group of unitary operators $U_t = \exp(-iAt)$, the subspaces $\mathfrak{D}_{\pm}$ have the following properties:

$$\begin{aligned} 1) \quad U_{\pm t} \mathfrak{D}_{\pm} \subset \mathfrak{D}_{\pm}, \quad t > 0; \quad 2) \quad \bigcap_t U_t \mathfrak{D}_{\pm} = \{0\}; \\ 3) \quad \bigcup_t U_t \mathfrak{D}_{\pm} = L_2(0, \infty; \mathfrak{N}); \quad 4) \quad \mathfrak{D}_+ \perp \mathfrak{D}_-. \end{aligned} \quad (18)$$

In view of (18), the group U_t is isomorphic to the groups of left shifts in the spaces $L_2(-\infty, \infty; \mathfrak{N}_{\pm})$. This isomorphism is established by isometric operators $F_{\pm}$, whose action on any function $f(x) \in L_2(0, \infty; \mathfrak{N})$ is defined by the formulas

$$(F_{\pm} f)(s) = P_{\pm}(U_s f)(x)|_{x=a}. \quad (19)$$

Here

$$F_{\pm} \mathfrak{D}_{\pm} = \{g(s) : g(s) \in L_2(-\infty, \infty; \mathfrak{N}_{\pm}), g(\pm s) = 0, s > 0\}.$$

To groups of unitary operators U_t for which there exist subspaces $\mathfrak{D}_{\pm}$ with properties (18), the scheme of scattering theory developed in recent works of P. Lax and R. Phillips for the study of wave equations [7] is applicable. The group U_t , generated by the differential operator (1) with a finite function $V(x)$, is thus another object to which this scheme is applicable.

6. We now abandon the assumption that the values of the function $V(x)$ are self-adjoint. Then, if equality (3) holds and

$$\int_0^{\infty} \|V(x)\| dx < \ln 2, \quad (20)$$

then, as before, the functions $\Phi(x, \lambda)$ ($\text{Im } \lambda = 0$) form a complete system in $L_2(0, \infty; \mathfrak{N})$, and the non-self-adjoint operator A , defined by expression (1) and condition (2), admits an exact definition analogous to that given above for the self-adjoint case.

Theorem 3. *If the function $V(x)$ satisfies conditions (3) and (20), then the group of operators $T_t = \exp(-iAt)$, $-\infty < t < \infty$, is uniformly bounded; there exist continuously invertible wave operators*

$$W_{\pm}(A, A_0) = s\text{-}\lim_{t \rightarrow \pm\infty} T_{-t} U_t^0,$$

where $U_t^0 = \exp(-iA_0 t)$ is the group of unitary operators generated by the operator A_0 considered; the groups of operators T_t and U_t^0 are similar, and

$$T_t = W_{\pm}(A, A_0) U_t^0 W_{\pm}^{-1}(A, A_0).$$

I express my gratitude to Prof. M. G. Krein for a valuable discussion.

Scientific Research Institute of Physics
of Odessa State University
named after I. I. Mechnikov

Received
27 II 1967

REFERENCES

1. M. G. Krein, DAN, 105, No. 3, 433 (1955).
2. M. G. Krein, DAN, 105, No. 4, 637 (1955).
3. M. G. Krein, DAN, 111, No. 6, 1167 (1956).
4. A. M. Rybalko, *Theory of Functions and Functional Analysis and Their Applications*, vol. 3, Kharkov, 1966, p. 42.
5. M. G. Gasymov, B. M. Levitan, DAN, 167, No. 5, 967, 1966.
6. M. G. Gasymov, B. M. Levitan, DAN, 167, No. 6, 1219 (1966).
7. R. D. Lax, R. S. Phillips, Bull. Am. Math. Soc., 70, 130 (1961).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.