

# INVESTIGATION OF SMALL SOLUTIONS OF A CLASS OF NONLINEAR OPERATOR EQUATIONS

V. M. Krasnosel'skii

1968

SovietRxiv

---

View the original and related papers at <https://sovietsrxiv.org/items/ru-196801.23540>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

UDC 517.948 + 513.88

*MATHEMATICS*

**V. M. KRASNOSEL' SKII**

# INVESTIGATION OF SMALL SOLUTIONS OF A CLASS OF NONLINEAR OPERATOR EQUATIONS

*(Presented by Academician N. N. Bogolyubov on 12 VI 1967)*

1. Let  $A$  be a nonlinear operator acting in a Banach space  $E$ . We shall assume that the operator  $A$  admits the representation

$$Ax = Bx + Cx + Dx, \quad (1)$$

where  $B$  is a linear operator,  $C$  is a quadratic operator, and  $D$  contains terms of third order of smallness:

$$\|Dx\| \leq p\|x\|^3.$$

We shall assume that the spectrum of the linear operator  $B$  consists of the point 1 and of some set  $F$  in the complex plane lying in the circle  $|z| \leq r_0 < 1$ . We shall suppose that the root subspace  $E_0$  corresponding to the eigenvalue 1 (see, for example, <sup>(1)</sup>) is two-dimensional and consists only of eigenvectors. Then, as is known, the space  $E$  can be represented as the direct sum of the subspace  $E_0$  and a certain subspace  $E^0$  invariant for  $B$ . The spectrum of the restriction  $B^0$  of the operator  $B$  to the subspace  $E^0$  coincides with the set  $F$  (see <sup>(2)</sup>). Every  $x \in E$  can be represented in the form  $x = x_0 + x^0$  ( $x_0 \in E_0$ ,  $x^0 \in E^0$ ); by  $P$  and  $Q$  we denote the projection operators defined by the equalities  $Px = x_0$ ,  $Qx = x^0$ .

Let  $r_0 < r_1 < 1$ . Since the spectrum of the restriction  $B^0$  of the operator  $B$  to the subspace  $E^0$  lies in the circle  $|z| \leq r_0$ , one can introduce in  $E^0$  such an equivalent norm  $\|x\|^0$  that

$$\|Bx\|^0 \leq r_1\|x\|^0 \quad (x \in E^0).$$

By  $\|x\|_0$  we denote some Euclidean norm in the plane  $E_0$ . Then on the entire space  $E$  the norm

$$\|x\|_* = \|Px\| + \|Qx\|^0 \quad (x \in E).$$

will be defined. This norm is equivalent to the original one.

Throughout the article it is assumed that the two-dimensional vector field  $PCx$  on the two-dimensional subspace  $E_0$  is nondegenerate; this means that  $PCx \neq 0$  if  $x \in E_0$  and  $x \neq 0$ . If in  $E_0$  one chooses some coordinate system, then the components of the field  $PCx$  will be quadratic forms in the coordinates of the point  $x$ .

We shall be interested in the problem of the arrangement in the space  $E$  of the set of small solutions of the equation

$$Ax = (1 + \mu)x \quad (2)$$

for small real  $\mu$ . This problem has been studied in detail by various methods for the case when 1 is a simple eigenvalue of the operator  $B$ . The results set forth below are obtained by methods of the theory of cones; closely related assertions can also be obtained by analytic methods.

2. An important role in our constructions will be played by the eigendirections of the operator  $PC$ . If, in some coordinate system  $\{\xi, \eta\}$ , the field  $PCx$  is given by the equality

$$PCx = \{a_1\xi^2 + 2b_1\xi\eta + c_1\eta^2; a_2\xi^2 + 2b_2\xi\eta + c_2\eta^2\}, \quad (3)$$

then the angular coefficient  $k$  of the eigendirections is determined from the cubic equation

$$c_1k^3 + (2b_1 - c_2)k^2 + (a_1 - 2b_2)k + a_2 = 0. \quad (4)$$

(if  $c_1 = 0$ , then the  $\eta$ -axis is an eigendirection of the operator  $PC$ ). The fundamental case is that in which all three roots of equation (4) are simple. This case is considered in the paper.

Denote by  $\mathfrak{N}(\mu_0, \rho_0)$  the set of nonzero solutions of equation (2) for  $|\mu| \leq \mu_0$ , lying in the ball  $\|x\| \leq \rho_0$ .

**Theorem 1.** *Let  $A$  be completely continuous. Then, for any  $\rho_0, \mu_0$ , the set  $\mathfrak{N}(\mu_0, \rho_0)$  is nonempty and forms a continuous branch in a neighborhood of zero (in the sense that on the boundary of each sufficiently small neighborhood of zero there is at least one point of the set  $\mathfrak{N}(\mu_0, \rho_0)$ ).*

Each eigendirection of the operator  $PC$  determines two eigenrays, one of which consists of eigenvectors corresponding to positive eigenvalues, and the other of eigenvectors corresponding to negative eigenvalues.

Denote by  $\Pi$  the union of all eigenrays  $l_i$  of the operator  $PC$ . If equation (4) has one simple real root, then  $\Pi$  consists of 2 rays; and if equation (4) has 3 real roots, then  $\Pi$  consists of 6 rays.

**Theorem 2.** *The set  $\mathfrak{N}(\mu_0, \rho_0)$  is tangent at zero to  $\Pi$  in the sense that*

$$\lim_{x \rightarrow 0, x \in \mathfrak{N}(\mu_0, \rho_0)} \frac{\rho(x, \Pi)}{\|x\|} = 0. \quad (5)$$

Equality (5) can be sharpened: the estimate  $\rho(x, \Pi) = O(\|x\|^{3/2})$  is valid. From the general theorems on bifurcation points (see, for example, <sup>(3,4)</sup>) there follows another important relation:  $\mu(x) \rightarrow 0$  as  $x \rightarrow 0$  ( $x \in \mathfrak{N}(\mu_0, \rho_0)$ ), where  $\mu(x)$  is the value of the parameter  $\mu$  for which  $x$  is a solution of equation (2).

3. Let  $K_i$  be a certain cone in the sense of M. T. Krein (see <sup>(5,6)</sup>), inside which lies the eigenray  $l_i$ . It follows from Theorem 2 that the small solutions of equation (2) from the set  $\mathfrak{N}(\mu_0, \rho_0)$  lie in the union of the cones  $K_i$ ; by  $\mathfrak{N}_i(\mu_0, \rho_0)$  we shall denote the intersection  $\mathfrak{N}(\mu_0, \rho_0) \cap K_i$ . The problem of small solutions of equation (2) will be studied if it is possible to describe sufficiently completely the small solutions lying in certain cones  $K_i$ .

Let  $l$  be a certain eigenray of the operator  $PC$ , and let  $h$  be the unit vector of this ray. Introduce into consideration the number

$$\beta(h) = \left. \frac{d}{d\varepsilon} (PC(h + \varepsilon g), -\varepsilon h + g) \right|_{\varepsilon=0}, \quad (6)$$

where  $g$  is a vector in  $E_0$  orthogonal to  $h$ ;  $\beta(h)$  is easily computed. Since the angular coefficient of the eigenray  $l$ , by assumption, is a simple root of equation (4),  $\beta(h) \neq 0$ . If  $\beta(h) < 0$ , then the eigendirection  $l$  will be called attracting, and if  $\beta(h) > 0$ , repelling.

Denote by  $K(l; \nu_1, \nu_2)$  ( $\nu_1, \nu_2 > 0$ ) the set of such  $x \in E$  that  $(Px, h) > 0$ ,  $|(Px, g)| \leq \nu_1(Px, h)$ ,  $\|Qx\|_* \leq \nu_2(Px, h)$ . It is easy to see that  $K(l; \nu_1, \nu_2)$  is a cone that admits plastering. Denote by  $T(l; \nu_1, \nu_2, \rho)$  the intersection of the cone  $K(l; \nu_1, \nu_2)$  with the ball  $\|x\|_* \leq \rho$ .

**Theorem 3** <sup>(7)</sup>. *Let the ray  $l$  be attracting. Then there exist positive numbers  $t_1, t_2, t_3, t_4$  such that, for every  $x \in T(l; \nu_1, \nu_2; \rho)$ , where  $\nu_1 \leq t_1$ ,  $\nu_2 \leq t_2\nu_1$ ,  $\rho \leq t_3\nu_2$ , the element  $Ax$  belongs to  $K(l; \nu_1, \nu_2)$  together with the ball consisting of elements of the form  $Ax + y$ , where  $\|y\|_* \leq t_4\nu_1\|x\|_*^2$ .*

4. Let us introduce into consideration the operator

$$A_1x = QAx - PAx + \frac{2(PA_1x, Px)}{(Px, Px)}Px. \quad (7)$$

This operator is defined for  $x \in E_0$ .

**Theorem 4.** *Let the ray  $l$  be repelling. Then there exist positive numbers  $t_1, t_2, t_3, t_4$  such that, for every  $x \in T(l; \nu_1, \nu_2; \rho)$ , where  $\nu_1 \leq t_1$ ,  $\nu_2 \leq t_2$ ,  $\rho \leq t_3 \nu_2$ , the element  $A_1 x$  belongs to  $K(l; \nu_1, \nu_2)$  together with the ball consisting of elements of the form  $A_1 x + y$ , where  $\|y\|_* \leq t_4 \nu_1 \|x\|_*^2$ .*

It is easy to verify that the solutions of equations (2) that do not lie in  $E^0$  coincide with the solutions of the equations  $A_1 x = (1 + \mu)x$ .

5. From Theorems 3 and 4 it follows that

**Theorem 5.** *Let the operator  $A$  be completely continuous. Then each set  $\mathfrak{R}_i(\mu_0, \rho_0)$  is nonempty and forms a continuous branch in a neighborhood of zero. Each continuous branch  $\mathfrak{R}_i(\mu_0, \rho_0)$  is tangent at the zero point to the corresponding eigenray  $l_i$ .*

In particular, from a simple analysis of equation (4) and from Theorem 5 it follows that:

**Theorem 6.** *Let the rotation <sup>(8)</sup> of the two-dimensional vector field (3) on the unit circle be equal to  $-2$ . Then  $\mathfrak{R}(\mu_0, \rho_0)$  consists of 6 continuous branches, each of which is tangent at zero to one of the 6 eigenrays of the quadratic operator  $PC$ .*

6. From the preceding arguments it follows that  $\mathfrak{R}(\mu_0, \rho_0)$  splits into either 2 or 6 branches  $\mathfrak{R}_i(\mu_0, \rho_0)$ . It can be shown that half of these branches consist entirely of solutions of equation (2) for positive  $\mu$ , and the other half of solutions of equation (2) for negative  $\mu$ .

7. The assertions given above can be applied, by the usual schemes, to the study of integral equations, problems on nonlinear oscillations, and others.

Voronezh State  
University

Received  
7 VI 1967

## REFERENCES

1. I. Ts. Gokhberg, M. G. Krein, *UMN*, **12**, no. 2, 43 (1957).
2. F. Riesz, B. Sz.-Nagy, *Lectures on Functional Analysis*, IL, 1954.
3. *Functional Analysis*, Reference Mathematical Library, "Nauka," 1964.
4. M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, 1956.

5. M. G. Krein, M. A. Rutman, *UMN*, **3**, no. 1 (23), 3 (1948).
6. M. A. Krasnosel' skii, *Positive Solutions of Operator Equations*, Moscow, 1962.
7. V. M. Krasnosel' skii, *Problems of the Mathematical Analysis of Complex Systems*, no. 1, Voronezh, 1967, p. 44.
8. P. S. Aleksandrov, *Combinatorial Topology*, 1948.

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*