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Abstract

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MATHEMATICS

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ON MAPPINGS CONNECTED WITH TOPOLOGICAL GROUPS

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The author regards this work as a continuation of the investigations of A. A. Markov ⁽¹⁾ and M. I. Graev ⁽²⁾ on topological groups. The results are stated; proofs are not given.

I. We shall call a space X : 1) **amoebic**, if there exists a disjoint covering of this space by open sets, each of which contains no more than one non-isolated point of the space X ; 2) **D -bicomact**, if it is a free union of bicompacts; 3) **standard**, if X is amoebic, D -bicomact, and metrizable simultaneously.

It is easy to see that condition 3) is equivalent to the following:

3') X is representable as a free union of spaces homeomorphic to a convergent sequence completed by its limit.

Theorem 1. *Every metrizable topological* group is representable as a quotient group of the free topological group of some standard space.*

Theorem 2. *If the space of a topological group is a k -space, then this group is isomorphic to a quotient group of the free topological group of some D -bicomact space.*

Theorem 3. *Every topological group is isomorphic to a quotient group of the free topological group of some amoebic space.*

The basis of Theorems 1, 2, and 3 is the following

Lemma 1. *Let $f : X \rightarrow Y$ be a quotient mapping onto, where X is a completely regular space and Y is the space of a topological group. Suppose further that $F(X)$ is the free topological group of the space X , and $\tilde{f} : F(X) \rightarrow Y$ is an extension of the mapping f to a continuous homomorphism of the group $F(X)$ onto the group Y . Then the mapping $\tilde{f}$ is open.*

Proof of the lemma. Denote by $\mathfrak{T}$ the topology given on Y , and by $\tilde{\mathfrak{T}}$ the quotient topology induced on Y by the topology of the group $F(X)$ via the mapping $\tilde{f}$. Then $\tilde{\mathfrak{T}} \supset \mathfrak{T}$. But the mapping $\tilde{f}$ of the space $F(X)$ onto Y , endowed with the topology $\tilde{\mathfrak{T}}$, is continuous; hence the mapping f of the space

X onto Y , endowed with the topology $\tilde{\mathfrak{T}}$, is also continuous. Since, with respect to the topology $\mathfrak{T}$, the mapping $f : X \rightarrow Y$ is a quotient mapping, $\mathfrak{T}$ contains every topology on Y with respect to which the mapping f is continuous. Hence, $\mathfrak{T} \supset \tilde{\mathfrak{T}}$ and $\tilde{\mathfrak{T}} = \mathfrak{T}$. The lemma is proved.

Three questions. Let X be a zero-dimensional D -bicomact space, or an amoebic space, or a standard space. Is it true that the free topological group of the space X is zero-dimensional in the sense of ind?

* We consider here only Hausdorff topological groups. By isomorphisms and quotient groups are meant topological isomorphisms and topological quotient groups.

Connected with them is the question:

Is every topological group whose space is a k -space representable as a quotient group of a zero-dimensional topological group?

The author does not know the answer to these questions, but he has proved

Theorem 4. *Every topological group with a countable base is a quotient group of some zero-dimensional topological group with a countable base.*

A more general assertion:

Theorem 4'. *If the space of a group is finally compact and paracompact, then this group is a quotient group of some zero-dimensional finally compact group.*

Theorem 5. *Every bicomact group is a quotient group of some zero-dimensional finally compact topological group (of the same weight).*

Corollary. *Every totally bounded group is a quotient group of some zero-dimensional (in the sense of ind) group.*

Probably these results should not be regarded as intuitively the most natural: A. Weil ⁽³⁾ stipulates that, on passing to a quotient group, the dimension cannot increase**, and C. Kaplan then constructs a counterexample (see ⁽³⁾, p. 170); we now see that this example is a manifestation of a general law.

The proofs of Theorems 4, 4', and 5 are based, in addition to Lemma 1 and the like, on the following assertions.

Proposition 1. *The free topological group of a zero-dimensional bicomactum is zero-dimensional.*

Proposition 2. *The free topological group of a zero-dimensional space with a countable base is zero-dimensional.*

The proof of the cited assertions is based on

General assertion 1. *Let X be a topological space, $F(X)$ the free topological group of this space, $F_n(X)$, where $n = 1, 2, \dots, \infty$, the set of all those elements*

of $F(X)$, i.e. those irreducible words, whose length does not exceed n . Put $A_n(X) = F_n(X) \setminus F_{n-1}(X)$, $n = 1, 2, \dots, \infty^{**}$. Then all $F_n(X)$ are closed in $F(X)$, and all natural mappings $A_n(X) \rightarrow (X \cup X^{-1})^n$ (where $(X \cup X^{-1})^n$ is the n -th Tikhonov power of the free union of two copies of the space X —for the notation see ⁽²⁾) are homeomorphisms into.*

The author does not know whether every metrizable group is a quotient group of a zero-dimensional metrizable group.

II. Here we touch upon questions of embedding topological spaces into spaces of groups. The well-known result of A. A. Markov is supplemented by

Theorem 6. *Every completely regular space of weight τ^{***} is homeomorphic to a closed subspace of a topological group whose weight does not exceed τ .**

Corollary. *Every regular space with a countable base is homeomorphic to a closed subspace of a topological group with a countable base.*

Our main goal in this section is to consider the question of the possibility of representing a bicom pactum in the form of a (closed) subset of type G_δ of some topological group. We shall call such representations $\aleph_0$ -embeddings of this bicom pactum.

The author's basic hypothesis: every bicom pactum admitting an $\aleph_0$ -embed-

* Zero-dimensionality may be understood both in the sense of ind and in the sense of dim.

** This is indeed so when the group is locally bicom pact (see ⁽⁴⁾).

*** $F_0(X)$ is the identity element of the group $F(X)$.

**** Here τ is any infinite cardinal number.

...embedding*, is a dyadic bicom pactum, remains unproved. But the following holds.

Theorem 7. *If $\mathcal{P}$ is a property of a topological space that is hereditary with respect to passage to a closed subspace, and every dyadic bicom pactum possessing the property $\mathcal{P}$ is metrizable, then every $\aleph_0$ -bicom pactum with the property $\mathcal{P}$ is also metrizable.*

The question of metrizability of dyadic bicom pacta has been well studied (see ^(6,7)); for example, we obtain the following corollaries:

Corollary 1. *If an $\aleph_0$ -bicom pactum satisfies the first axiom of countability or, more generally, is a sequential space—the latter being equivalent to the assertion that this $\aleph_0$ -bicom pactum is a quotient space of a metric space—then it is metrizable.*

Corollary 2. *If every closed subspace of some $\aleph_0$ -bicom pactum is an $\aleph_0$ -bicom pactum, then this $\aleph_0$ -bicom pactum is metrizable.*

Let us note that each of the conditions formulated so far that are sufficient for metrizable of an $\aleph_0$ -bicom pactum is a necessary condition for metrizable of any bicom pactum. The following is of a different nature.

Theorem 8. *The set of extremally disconnected points of an $\aleph_0$ -bicom pactum is finite (or empty).*

Theorems 7 and 8 are proved on the basis of the following considerations. Let $\varphi : G \rightarrow H$ be a continuous homomorphism from one topological group onto another, let $Y \subset H$ be some subspace, and let $X = \varphi^{-1}Y$ be a subspace of the group G . By f we denote the continuous mapping $X \rightarrow Y$ induced by φ .

Any mapping of one space X onto another topological space which can be represented, in the manner indicated above, by means of some open homomorphism φ , we shall call a **skew mapping**.

Theorem 9. *Every $\aleph_0$ -bicom pactum can be skew mapped onto a compactum.*

Let us note that a skew mapping is always open, and that the inverse images of points under it are homeomorphic to one another and to some group.

Proposition 3. *If the inverse images of points under a skew mapping of a bicom pactum X onto a compactum Y are compacta, then X is metrizable.*

For the proof of the last assertion we need the following

Lemma. *If the space of a group has a countable pseudocharacter, then every bicom pactum lying in it is metrizable.*

The general features of the structure of nonmetrizable $\aleph_0$ -bicom pacta are revealed by

Theorem 10. *Every nonmetrizable $\aleph_0$ -bicom pactum is homogeneous with respect to character and weight, and the weight of the whole bicom pactum is equal to its character at an arbitrary point.*

Theorem 11. *The Suslin number of an arbitrary $\aleph_0$ -bicom pactum does not exceed $\aleph_0$.*

Theorem 12. *A bicom pactum is $\aleph_0$ -embedded in its free topological group if and only if it is metrizable.*

Hence (or from Theorem 6) it follows that

Corollary. *Every compactum is an $\aleph_0$ -bicom pactum.*

On the other hand, it follows from Theorem 10 that not every dyadic bicom pactum is an $\aleph_0$ -bicom pactum; consequently, the property of being an $\aleph_0$ -bicom pactum is not invariant under arbitrary continuous mappings (invariance under open mappings is possible).

Theorem 13. *If the cardinality of an $\aleph_0$ -bicom pactum is less than $2^{\aleph_1}$, then it is metrizable.*

* In what follows we call bicom pacta admitting $\aleph_0$ -embeddings $\aleph_0$ -bicom pacta.

Assuming the continuum hypothesis, we conclude that every $\aleph_0$ -bicom pactum of cardinality continuum is metrizable.

Theorem 14. *If a bicom pactum is $\aleph_0$ -embedded in the space of a completely bounded topological group, then it is dyadic.*

In connection with Theorems 9, 12, and 14, and the hypothesis on the dyadicity of an arbitrary $\aleph_0$ -bicom pactum, the following question is of interest:

Question. Is the preimage of a compactum under a bicom pact homomorphism necessarily a dyadic bicom pactum?

III. Questions concerning the location of bicom pacta lead to the consideration of bicom pactly generated groups.*

Theorem 15. *If a bicom pactly generated group is extremally disconnected, then it is countable and finitely generated.*

Theorem 15 is based on

Theorem 16. *Every bicom pact subset of an extremally disconnected** topological group is finite (or empty).*

It remains unclear whether there exists a nondiscrete finitely generated extremally disconnected topological group. The space of the latter, by Theorem 15, cannot be a k -space.

Theorem 17. *If a bicom pact space with the first axiom of countability can be topologically embedded in some bicom pact group as a set of algebraic generators of the latter, then both this space and the group are metrizable.*

Corollary. *Not every bicom pactum can be represented as an algebraically generating subspace of a bicom pact group.*

As is known, every bicom pactum can be represented as the set of algebraic generators of some completely bounded group and can be embedded in a bicom pact group. This means that the last assertions are of a definitive character.

Lemma 2. *If a topological group X is algebraically generated both by a subspace Y and by a bicom pact subspace Φ , then $\text{weight } \Phi \leq \text{weight } Y^{**}$.**

Theorem 18. *If a topological group X is algebraically generated by its bicom pact subspaces Φ_1 and Φ_2 , then the weight of Φ_1 is equal to the weight of Φ_2^{**} .**

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* Groups are meant which are algebraically generated by their bicomact subspace.

** A more general assertion: if, in the space of a topological group, the closures of any two disjoint open subsets of type F_σ do not intersect, then every bicomactum lying in it is finite. I am grateful to W. W. Comfort, who drew my attention to the named class of spaces.

*** If these weights are infinite.

Note: Figure translations are in progress. See original paper for figures.

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