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E. T. Ivlev

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Abstract

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MATHEMATICS

E. T. IVLEV

ON A MULTIDIMENSIONAL SURFACE IN PROJECTIVE SPACE

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One of the fundamental problems of the differential geometry of a multidimensional surface is the problem of invariant equipment. A sufficiently complete survey of works on this problem was given by G. F. Laptev in ⁽¹⁾ (see also ⁽²⁻¹⁰⁾). As G. F. Laptev notes, following the problem of the invariant equipment of a multidimensional surface there arises the problem of constructing a canonical frame. In the present paper a canonical frame is constructed for an m -parametric manifold $S_m(l_\alpha)$, consisting of m -surfaces S_m , through each point S of which pass m lines Γ_α belonging to it, with tangents l_α in general position. It is assumed here that $2 < m < n - 2$. The canonical frame of the manifold $S_m(l_\alpha)$ is a semicanonical frame in the sense of ^(14,16) of an arbitrary m -surface S_m in P_n .

We shall write the derivation formulas of a certain frame $\{A_i\}$ of the manifold $S_\alpha(l_m)$ in P_n in the form $dA_i = \omega_i^k A_k$ ($i, k = 0, 1, \dots, n$), where ω_i^k are Pfaffian differential forms satisfying the structure equations $D\omega_i^k = [\omega_i^j \omega_j^k]$ ($j = 0, 1, \dots, n$) and the relation $\omega_i^i = 0$. Put $k_a = m_a - m_{a-1} = C_{m+a}^{a+1}$. Introduce the system of indices $\alpha, \beta, \gamma, \alpha_0, \beta_0, \gamma_0, \alpha^q, \beta^q, \gamma^q$ ($q \geq 1$) = $1, 2, \dots, m$; $\alpha_a, \beta_a, \gamma_a, \alpha_a^q, \beta_a^q, \gamma_a^q = m_{a-1} + 1, \dots, m_a$ ($a, b, c = 1, \dots, p$; $m = m_0$); $\alpha_p, \beta_p, \gamma_p = m_{p-1} + 1, \dots, n$; $\alpha_{a \dots b} = m_{a-1} + 1, \dots, m_a, \dots, m_{b-1} + 1, \dots, m_b$ ($a \leq b$); $\alpha_{a \dots a} = \alpha_a$; $\alpha, \beta, \gamma = m + 1, \dots, n$. Let $A_0 = S$, $l_\alpha = (A_0 A_\alpha)$. Then the systems of differential equations $\omega_0^\alpha = 0$ and $\omega_\alpha^\beta = \Lambda_{\alpha\gamma}^\beta \omega^\gamma$ ($\alpha \neq \beta$) determine the m -surface S_m and the system of lines Γ_α on it, respectively. Successive prolongations of the system $\omega_0^\alpha = 0$ lead to a sequence of fundamental objects $\{\Lambda_{\alpha^1 \alpha^2 \dots \alpha^r}^{\hat{\alpha}}\}$ ⁽¹⁷⁾, whose components are symmetric with respect to the lower indices and satisfy the system of differential equations ^(1,3) in ⁽⁴⁾. The fixing of the frame, carried out analytically, gives

$$\Lambda_{\alpha^1 \alpha^2 \dots \alpha^{b+1}}^{b+1 \dots p} = 0, \quad \bar{b}_{\beta_a}^{\alpha_a} = \delta_{\beta_a}^{\alpha_a}, \quad \Lambda_{0\beta}^{\hat{\alpha}} = 0, \quad \Lambda_{0\beta}^\alpha = \delta_\beta^\alpha,$$

$$\Lambda_{12}^\tau = \Lambda_{m-1,m}^1 = \Lambda_{m-1,m}^2 = 1, \quad \Lambda_{0\alpha}^0 = 0, \quad \Lambda_{(\alpha\gamma)}^\beta = 0, \quad \Lambda_{i\alpha}^i = 0,$$

$$\Lambda_{\alpha_1 \dots p-2\beta}^{\alpha} = 0, \quad \Lambda_{\alpha-1\beta}^{\alpha+1-p} = 0, \quad \Lambda_{\alpha_1 \dots p-2\beta}^0 = 0,$$

$$\Lambda_{\alpha_{p-2}[\alpha] \Lambda_{1\beta_{p-1}|\beta]}^{\alpha_{p-1}} = 0, \quad \Lambda_{\alpha_{p-2}[\alpha] \Lambda_{1\beta_{p-1}|\beta]}^{\alpha_{p-1}} = 0, \quad \Lambda_{\beta_{f+1} \dots g\alpha}^{\alpha_f} = 0,$$

$$\Lambda_{\alpha^1 \dots \alpha^{f+1}}^{\beta_f} \Lambda_{\beta_f \alpha}^{\alpha_f} = \frac{1}{f!} \Lambda_{\sigma(\alpha^1 \dots \alpha^f \Lambda_{\alpha^{f+1}}^{\sigma})\alpha}^{\alpha_f}, \quad \Lambda_{\alpha_{p-2}[\alpha] \Lambda_{1\beta_{p-1}|\beta]}^{\alpha_{p-1}} = 0,$$

$$\Lambda_{\alpha_{p-1}}^{\beta_f} = 0 \quad (n \leq m_{p-1} + k_{p-1}); \quad \Lambda_{\alpha_{p-1}}^{\beta_f} = \Lambda_{\alpha_{p-1}}^{\beta_{f+1}} = 0$$

$$(q(p-1, s) < n \leq q(p-1, s+1)).$$

Here and in what follows the following notation and numbering of indices are adopted:

- 1) $\bar{b}_{\beta_a}^{\alpha_a} = \Lambda_{\alpha^1 \dots \alpha^{a+1}}^{\alpha_a}$, where the system of quantities $\Lambda_{\alpha^1 \dots \alpha^{a+1}}^{\alpha_a}$, symmetric in any pair of lower indices, for each fixed upper index consists of k_a independent components $\Lambda_{\alpha^1 \dots \alpha^{a+1}}^{\alpha_a}$, in which to each value of the collection of indices there is assigned a definite value of the index β_a , i.e. $\beta_a \leftrightarrow (\alpha^1 \dots \alpha^{a+1})$, and the indices $\alpha^1, \dots, \alpha^{a+1}$ are arranged successively so that each subsequent index is not less than the preceding one;
- 2) $f = 1, 2, \dots, p-1$; $g = f+1, \dots, p$;
- 3) $\alpha_{p-1}^0 = m_{p-2} + 1, \dots, m_{p-2} + n - m_{p-1}$; $\beta_p^0 \leftrightarrow (\alpha_{p-1}^0 1)$, $\alpha_{p-1}^0 \leftrightarrow (\alpha^1 \dots \alpha^p)$, $\beta_p^0 = m_{p-1} + 1, \dots, n$ (in the case $m_{p-1} < n \leq m_{p-1} + k_{p-1}$);
- 4) $\alpha_{p-1}^\nu = q(p-2, \nu) + 1, \dots, q(p-2, \nu+1)$, $\alpha_{p-1}^s = q(p-2, s) + 1, \dots, \dots, m_{p-2} + n - q(p-1, s)$, $\beta_p^\nu \leftrightarrow (\alpha_{p-1}^\nu, \nu+1)$, $\alpha_{p-1}^\nu \leftrightarrow (\alpha^1 \dots \alpha^p)$, $\beta_p^\nu = m_{p-1} + k_{p-1} + 1, \dots, n$,
 $\beta_p^* \leftrightarrow (\alpha_{p-1}^* 1)$, $\alpha_{p-1}^* \leftrightarrow (\alpha^1 \dots \alpha^p)$, $\alpha_{p-1}^* = m_{p-2} + 1, \dots, m_{p-1}$;
 $\beta_p^* = m_{p-1} + 1, \dots, m_{p-1} + k_{p-1}$,

where

$$q(r, t) = m_r + k_r + \dots + k_r^{t-1}, \quad k_r^s = C_{m+r-s}^{r+1}, \quad k_r^0 = k_r$$

$$(s = 1, \dots, m-1, s \text{ fixed}; t = 1, 2, \dots; r = p-2, p-1);$$

- 5) $r = 3, \dots, m$.

The fixation of the canonical frame of the manifold $S_m(l_a)$, carried out by formulas (1), has the following geometric interpretation. The linear subspace $L_m = (A_0 A_1 \dots A_m)$ is the tangent m -plane of the surface S_m at the point A_0 , while $dL_{m_a} = (L_{m_{a-1}} A_{m_{a-1}+1} \dots A_{m_a})$ is the osculating m_a -plane of order $a+1$ of the surface S_m at the point A_0 in the sense of (11). Each (m_a+1) -plane

$$L_{m_{a-1}+1}^{\alpha_a} = (L_{m_{a-1}} A_{\alpha_a})$$

from L_{m_a} contains, along the corresponding coordinate line Γ_a of the surface S_m , the first differential neighborhood of the corresponding $(m_{a-1}+1)$ -plane

$$L_{m_{a-2}+1}^{\alpha_{a-1}} = (L_{m_{a-3}} A_{\alpha_{a-1}})$$

from $L_{m_{a-2}}$ (for $a=1$ the role of $L_{m_{-1}+1}^{\alpha_1}$ is played by the lines $A_0 A_{\alpha_1}$). The linear subspace

$$L_{m-1}^* = (A_1 A_2 \dots A_m)$$

is a normal of the first kind in the sense of (12), and the k_1 -plane

$$L_{k_1}^* = (A_0 A_{m+1} \dots A_{m_1})$$

is chosen so that the space with connection, in the sense of (12, 13), induced by the k_1 -plane $L_{k_1}^*$ along S_m , whose geometric element (17) is the point A_0 and the m -plane L_m , is an affine homogeneous space. At the same time the manifold

$$X_m^{m-1}$$

is an m -parameter manifold of normals of the second kind—harmonically conjugate to the surface S_m in the sense of (12). Each (m_a-1) -plane

$$L_{m_a-1}^* = (L_{m_{a-1}-1}^* A_{m_{a-1}+1} \dots A_{m_a}) \quad (a=1, \dots, p-1)$$

is an osculating linear subspace of order a of the manifold X_m^{m-1} in the element L_{m-1}^* , i.e. it contains all differential neighborhoods of the $(m-1)$ -plane L_{m-1}^* up to order a inclusive. Consequently,

$$l_{m_b}^{\alpha_{b+1}} = (L_{m_b-1}^* A_{\alpha_{b+1}}) = L_{m_{b+1}}^{b+1} \cap L_{m_b-1}^* \quad (b=1, \dots, p-2; L_{m_{b+1}}^{\alpha_{b+1}} = (L_{m_b-1} A_{\alpha_{b+1}})).$$

In the case $m_{p-1} < n \leq m_{p-1} + k_{p-1}$, each linear subspace

$$L_{m_{p-1}}^{*\beta_p^0} = (L_{m-1}^* A_{\beta_p^0}),$$

belonging to $L_{m_{p-1}+1}^{\beta_p^0}$ and not passing through the point A_0 , contains the first differential neighborhood of the m_{p-2} -plane $L_{m_{p-2}}^{p-1}$ under the displacement $\omega^2 = \dots = \omega^m = 0$. Analogously characterized are

geometrically the m_{p-1} -planes $L_{m_{p-1}}^{*\beta_p^\nu} = (L_{m_{p-1}-1}^* A_{\beta_p^\nu})$ and $L_{m_{p-1}}^{*\beta_p^*} = (L_{m_{p-1}-1}^* A_{\beta_p^*})$ in the case $q(p-1, s) < n \leq q(p-1, s+1)$. Thus the hyperplane $L_{n-1} = (A_1 A_2 \dots A_n)$ is geometrically determined by the fact that it passes through all $L_{m_b}^{*\alpha_{b+1}}$ and $L_{m_{p-1}}^{*\alpha_p}$.

Each linear subspace $L_{k_1+\dots+k_a}^{*\alpha_{a+1}} = (L_{k_1+\dots+k_a-1}^* A_{\alpha_{a+1}})$ is such that

$$L_{k_1+\dots+k_a}^{*\alpha_{a+1}} = L_{m_{a+1}}^{\alpha_{a+1}} \cap L_{k_1+\dots+k_{a+1}}^*,$$

where $L_{k_1+\dots+k_{a+1}}^*$ is the tangent linear subspace of order a of the variety $X_m^{k_1-1}$ —the m -parameter variety of (k_1-1) -planes $L_{k_1-1}^* = L_{k_1} \cap L_{m_1-1}^*$. In the case $m_{p-1} < n \leq m_{p-1} + k_{p-1}$, every $(m_{p-1} - m)$ -plane

$$L_{k_1+\dots+k_{p-1}}^{*\beta_p^0} = (L_{k_1+\dots+k_{p-1}-1}^* A_{\beta_p^0})$$

contains the first differential neighborhood of the linear subspace $L_{k_1+\dots+k_{p-2}}^{*\alpha_{p-1}^0}$ under a displacement $\omega^2 = \omega^3 = \dots = \omega^m = 0$. The $(m_{p-1} - m)$ -planes $L_{k_1+\dots+k_{p-1}}^{*\beta_p}$ and $L_{k_1+\dots+k_{p-1}}^{*\beta_p^v}$ are characterized analogously in the case $q(p-1, s) < n \leq q(p-1, s+1)$. Thus the normal of the first kind in the sense of (12)—the $(n-m)$ -plane

$$L_{n-m} = (A_0 A_{m+1} \dots A_n)$$

—is geometrically characterized by the fact that it passes through the point A_0 and all linear subspaces $L_{k_1+\dots+k_a}^{*\alpha_{a+1}}$ ($a = 1, \dots, p-2$) and $L_{k_1+\dots+k_{p-1}}^{*\alpha_p}$. Since the normals of the first and second kind are geometrically determined, the surface S_m is normalized in the sense of A. P. Norden (12). Each linear subspace

$$L_{k_{b-1}+k_b-1}^* = (L_{k_{b-1}-1}^* A_{m_b+1} \dots A_{m_{b+1}})$$

is chosen so that it induces, along $X_m^{k_{b-1}-1}$ —the m -parameter variety of linear subspaces $L_{k_{b-1}-1}^* = (A_{m_{b-2}+1} \dots A_{m_{b-1}})$ ($b = 2, \dots, p-1$)—a space with connection that is an affine homogeneous space. Consequently,

$$L_{k_{b+1}-1}^* = (A_{m_b+1} \dots A_{m_{b+1}}) = L_{k_{b+1}+k_{b-1}-1}^* \cap L_{k_1+\dots+k_{b+1}-1}^*.$$

Each $(k_{b+1} + \dots + k_a - 1)$ -plane

$$L_{k_{b+1}+\dots+k_a-1}^* = (L_{k_{b+1}+\dots+k_{a-1}-1}^* A_{m_{a-1}+1} + \dots + A_{m_a})$$

($b = 1, \dots, p-1$; $a = b+1, \dots, p$) is a tangent linear subspace of order $a-b$ of the variety $X_m^{k_{b+1}-1}$ —the m -parameter variety of linear subspaces $L_{k_{b+1}-1}^*$. Therefore

$$L_{k_{b+1}+\dots+k_a}^{\alpha_a} = (L_{k_{b+1}+\dots+k_{a-1}-1}^* A_{\alpha_a}) = L_{k_{b+1}+\dots+k_{a-1}}^{\alpha_a} \cap L_{m_{a-1}+1}^{\alpha_a}.$$

Each $(k_{b+1} + \dots + k_{p-1})$ -plane

$$L_{k_{b+1}+\dots+k_{p-1}}^{*\beta_p^0} = (L_{k_{b+1}+\dots+k_{p-1}-1}^* A_{\beta_p^0})$$

in the case $m_{p-1} < n \leq m_{p-1} + k_{p-1}$ contains the first differential neighborhood of the $(k_{b+1} + \dots + k_{p-2})$ -plane

$$L_{k_{b+1}+\dots+k_{p-2}}^{*\alpha_{p-1}^0} = (L_{k_{b+1}+\dots+k_{p-2}-1}^* A_{\alpha_{p-1}^0})$$

under a displacement $\omega^2 = \dots = \omega^m = 0$. The $(k_{b+1} + \dots + k_{p-1})$ -planes $L_{k_{b+1}+\dots+k_{p-1}}^{*\beta_p}$ and $L_{k_{b+1}+\dots+k_{p-1}}^{*\beta_p^\nu}$ are also determined geometrically in the case $q(p-1, s) < n \leq q(p-1, s+1)$. Each k_{p-1} -plane

$$L_{k_{p-1}}^{*\beta_p^0} = (L_{k_{p-1}-1}^* A_{\beta_p^0})$$

in the case $m_{p-1} < n \leq m_{p-1} + k_{p-1}$ contains the tangent to

line described by the point $A_{\alpha_{p-1}^0}$ under the displacement $\omega^2 = \dots = \omega^m = 0$.

Analogously, the k_{p-1} -planes $L_{k_{p-1}}^{*\beta_p^*} = (L_{k_{p-1}}^* A_{\beta_p^*})$ and $L_{k_{p-1}}^{*\beta_p^0} = (L_{k_{p-1}}^* A_{\beta_p^0})$ are geometrically defined in the case $q(p-1, s) < n \leq q(p-1, s+1)$. In the case $m_{p-1} < n \leq k_{p-1} + m_{p-1}$, in the linear subspace there exists such a point $A_{\beta_p^0}$ for each β_p^0 that the line $A_{\alpha_{p-1}^0} A_{\beta_p^0}$ is tangent to the line described by the point $A_{\alpha_{p-1}^0}$ under the displacement $\omega^2 = \dots = \omega^m = 0$. The points $A_{\beta_p^*}$ and $A_{\beta_p^0}$ are analytically defined in the case $q(p-1, s) < n \leq q(p-1, s+1)$. Thus, the points A_α belong simultaneously to the lines $l_\alpha = (A_0 A_\alpha)$ and to the normal of the second kind, while each point A_{α_α} belongs simultaneously to the $(m_\alpha + 1)$ -plane $L_{m_{\alpha-1}+1}^{\alpha_\alpha} = (L_{m_{\alpha-1}} A_{\alpha_\alpha})$ and to the $(k_\alpha - 1)$ -plane $L_{k_{\alpha-1}}^* = (A_{m_{\alpha-1}+1} \dots A_{m_\alpha})$. Therefore all elements of the canonical frame of the manifold $S_m(l_\alpha)$ are geometrically characterized. In this frame we single out the following two special classes: 1) the manifold $S_m(l_\alpha)$, determined by the relations $\Lambda_{\alpha\beta}^\beta = 0$ (summation over β), exists with an arbitrariness of $n + m(m-3)$ functions of m arguments and is characterized by the fact that the points A_α on the lines $A_0 A_\alpha$ are harmonic poles⁽¹⁸⁾ of the point A_0 with respect to the pseudofoci F_α^β ($\alpha \neq \beta$)⁽¹⁵⁾; 2) the manifold $S_m(l_\alpha)$, determined by the natural equations $\Lambda_{\alpha\beta}^\beta = 0$ ($\alpha \neq \beta$, with no summation over β), exists with an arbitrariness of $n-m$ functions of m arguments and is characterized by the fact that all pseudofoci F_α^β of each line $A_0 A_\alpha$ coincide with the point A_α .

Tomsk State University
named after V. V. Kuibyshev

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