

AN OPERATIONAL METHOD FOR SOLVING MIXED PROBLEMS FOR LINEAR DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE WITH DISCONTINUOUS COEFFICIENTS

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Abstract

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MATHEMATICS

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AN OPERATIONAL METHOD FOR SOLVING MIXED PROBLEMS FOR LINEAR DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE WITH DISCONTINUOUS COEFFICIENTS

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The mixed problem (1)–(3) was investigated in the general case of complex-valued functions $c_2^{(i)}(x)$, $-\pi/2 < \arg c_2^{(i)}(x) < \pi/2$ ($i = 1, \dots, n$), by the contour-integral method in the monograph of M. L. Rasulov (¹). We consider an operational scheme of solution and give a justification of the operational method for solving problem (1)–(3) on the half-line $t > 0$. The realization of the obtained operational formulas of solution (8) is connected with the inversion of Laplace integrals. Numerical methods for inverting these integrals have been sufficiently developed (⁵, Ch. 5).

1. Consider the problem of finding solutions of the equations

$$\frac{\partial u^{(i)}(x, t)}{\partial t} = \sum_{l=0}^2 c_l^{(i)}(x) \frac{\partial^l u^{(i)}(x, t)}{\partial x^l} + f^{(i)}(x, t), \quad (1)$$

$$x \in (a_i, b_i), \quad t \in (0, T], \quad (i = 1, \dots, n)$$

under the boundary conditions

$$\sum_{i=1}^n \sum_{l=0}^1 \left\{ \alpha_{sl}^{(i)} \left(\frac{\partial}{\partial t} \right) \frac{\partial^l u^{(i)}(x, t)}{\partial x^l} \Big|_{x=a_i} + \beta_{sl}^{(i)} \left(\frac{\partial}{\partial t} \right) \frac{\partial^l u^{(i)}(x, t)}{\partial x^l} \Big|_{x=b_i} \right\} = \gamma_s, \quad (2)$$

$$t \in (0, T], \quad (s = 1, 2, \dots, 2n)$$

and the initial conditions

$$u^{(i)}(x, +0) = 0, \quad x \in (a_i, b_i) \quad (i = 1, \dots, n), \quad (3)$$

where

$$\alpha_{sl}^{(i)} \left(\frac{\partial}{\partial t} \right) = \alpha_{sl0}^{(i)} + \alpha_{sl1}^{(i)} \frac{\partial}{\partial t}, \quad \beta_{sl}^{(i)} \left(\frac{\partial}{\partial t} \right) = \beta_{sl0}^{(i)} + \beta_{sl1}^{(i)} \frac{\partial}{\partial t},$$

$\alpha_{slk}^{(i)}, \beta_{slk}^{(i)}$ ($l, k = 0, 1$), γ_s ($s = 1, 2, \dots, 2n$) are, generally speaking, complex constants; (a_i, b_i) are mutually nonintersecting intervals having common boundary points and contained in the basic interval (a_1, b_n) ($n \geq 1$).

Problem (1)–(3) is reduced to two problems: **problem A**—the determination of solutions $v^{(i)}(x, t)$ of the homogeneous equations corresponding to equation (1), under the boundary conditions (2) and the initial conditions (3), and **problem B**—the determination of solutions $w^{(i)}(x, t)$ of equations (1) satisfying the homogeneous boundary conditions obtained from (2) when $\gamma_s = 0$ ($s = 1, 2, \dots, 2n$), and the initial conditions (3).

By a solution of problems A and B we shall mean functions $v^{(i)}(x, t)$ and $w^{(i)}(x, t)$ ($i = 1, \dots, n$) that are continuous for $x \in (a_i, b_i)$, $t \in [0, T]$, and for $x \in [a_i, b_i]$, $t \in (0, T]$ ($0 < T < \infty$), all derivatives entering into (1) are continuous with respect to $x \in (a_i, b_i)$ and $t \in (0, T]$, while the derivatives entering into (2) are continuous with respect to $x \in [a_i, b_i]$ and $t \in (0, T]$.

We shall assume the following conditions to be fulfilled:

- a) the functions $c_l^{(i)}(x)$ ($l = 0, 1, 2$) are $6-l$ times continuously differentiable on $[a_i, b_i]$, with $c_0^{(i)}(x)$, $c_1^{(i)}(x)$ complex-valued, and $c_2^{(i)}(x)$ real-valued functions, and $c_2^{(i)}(x) > 0$ for all $x \in [a_i, b_i]$;
- b) for any $t \in [0, T]$ the functions $\partial^k f^{(i)}(x, t) / \partial t^k$ ($k = 0, 1, 2$) and $\partial f^{(i)}(x, t) / \partial x$ are continuously differentiable with respect to x on $[a_i, b_i]$; $f^{(i)}(x, t)$ are complex-valued functions.

Taking for $z = \sqrt{\lambda}$ that branch for which $z \in \tilde{\Pi}_1 \{z : 0 \leq \arg z \leq \pi/2\}$ when $\lambda \in \Pi_1 \{\lambda : 0 \leq \arg \lambda \leq \pi\}$ and $z \in \tilde{\Pi}_2 \{z : -\pi/2 \leq \arg z \leq 0\}$ when $\lambda \in \tilde{\Pi}_2 \{\lambda : -\pi \leq \arg \lambda \leq 0\}$, put

$$A_{sk}^{(i)}(z) = \sum_{l=0}^1 \alpha_{sl}^{(i)}(z^2) (z \varphi_k^{(i)}(a_i))^l, \quad B_{sk}^{(i)}(z) = \sum_{l=0}^1 \beta_{sl}^{(i)}(z^2) (z \varphi_k^{(i)}(b_i))^l, \quad \varphi_k^{(i)}(x) = (-1)^k (c_2^{(i)}(x))^{-1/2} \quad (k = 1, 2; s = 1, 2, \dots, 2n).$$

Denoting by $A_k^{(i)}(z), B_k^{(i)}(z)$ ($k = 1, 2$) the columns with elements $A_{1k}^{(i)}(z), \dots, A_{2n,k}^{(i)}(z)$ and $B_{1k}^{(i)}(z), \dots, B_{2n,k}^{(i)}(z)$, respectively, we formulate conditions A and B_(k) ($k = 0, 1, 2$).

A. The determinants

$$\left| A_1^{(1)}(z) B_2^{(1)}(z) A_1^{(2)}(z) B_2^{(2)}(z) \dots A_1^{(n)}(z) B_2^{(n)}(z) \right|,$$

$$\left| B_1^{(1)}(z) A_2^{(1)}(z) B_1^{(2)}(z) A_2^{(2)}(z) \dots B_1^{(n)}(z) A_2^{(n)}(z) \right|$$

are polynomials in z of the same degree $d \geq 0$, different from the identically zero polynomial. All determinants of order $2n$ formed from other combinations of the columns $A_k^{(i)}(z), B_k^{(i)}(z)$ ($k = 1, 2; i = 1, \dots, n$) are polynomials in z of degree not exceeding d .

B_(k). Suppose that in condition A $d \geq k$ and all determinants of order $2n - 1$ formed from the elements of the matrix

$$(A_1^{(1)}(z) A_2^{(1)}(z) \dots A_1^{(n)}(z) A_2^{(n)}(z) B_1^{(1)}(z) B_2^{(1)}(z) \dots B_1^{(n)}(z) B_2^{(n)}(z))$$

are polynomials in z of degree not exceeding $d - k$.

To the original problem (1)–(3) we assign the boundary-value problem with a complex parameter (the spectral problem)

$$\sum_{l=0}^2 c_l^{(i)}(x) \frac{d^l y^{(i)}}{dx^l} - \lambda y^{(i)} = 0, \quad a_i < x < b_i \quad (i = 1, \dots, n); \quad (4)$$

$$\sum_{i=1}^n \sum_{l=0}^1 \left\{ \alpha_{sl}^{(i)}(\lambda) \frac{d^l y^{(i)}}{dx^l} \Big|_{x=a_i} + \beta_{sl}^{(i)}(\lambda) \frac{d^l y^{(i)}}{dx^l} \Big|_{x=b_i} \right\} = \gamma_s \quad (s = 1, 2, \dots, 2n). \quad (5)$$

Let $y_k^{(i)}(x, \lambda)$ ($k = 1, 2$) be a fundamental system of particular solutions of the i -th equation (4), consisting of functions which, together with their first derivatives with respect to x , are entire functions of the parameter λ (see Poincaré' s theorem ⁽⁴⁾, p. 28). Denote

$$u_{sk}^{(i)}(\lambda) = \sum_{l=0}^1 \left\{ \alpha_{sl}^{(i)}(\lambda) \frac{d^l y_k^{(i)}}{dx^l} \Big|_{x=a_i} + \beta_{sl}^{(i)}(\lambda) \frac{d^l y_k^{(i)}}{dx^l} \Big|_{x=b_i} \right\}$$

$$(k = 1, 2; i = 1, \dots, n; s = 1, 2, \dots, 2n).$$

Now making use of the auxiliary notation from ⁽¹⁾, we introduce for consideration, by formulas (6.2.5), (6.2.12)–(6.2.16) of ⁽¹⁾, the functions $w^{(i)}(x, \lambda)$,

$\Delta^{(i)}(x, \lambda)$, $\Delta^{(i,j)}(x, \xi, \lambda)$, $g^{(i,j)}(x, \xi, \lambda)$, $\Delta(\lambda)$, $G^{(i,j)}(x, \xi, \lambda)$. Following Birkhoff⁽³⁾, we shall denote a sum of the form $f(x) + E(x, z)/z$, where $E(x, z)$ is continuous in x and bounded for $|z| \geq R$ (R is a sufficiently large positive number), by the symbol $[f(x)]$. Then, applying Tamarkin's theorem on the asymptotic representation of a certain fundamental system of particular solutions to each i -th equation of (4) (see⁽²⁾, theorem 3), for the determinant $\tilde{\Delta}(z) = \Delta(z^2)$ we find, for all sufficiently large in modulus $z \in \tilde{\Pi}$, $\tilde{\Pi} = \tilde{\Pi}_1 + \tilde{\Pi}_2$, the asymptotic formula

$$\tilde{\Delta}(z) = z^{d-n} H(z) \prod_{i=1}^n \psi^{(i)}(a_i z^2),$$

$$H(z) = \sum_{\mu=1}^{\sigma} [M_{\mu}] \exp(m_{\mu} z), \quad m_1 < \dots < m_{\sigma},$$

$$-m_1 = m_{\sigma} = \sum_{i=1}^n \int_{a_i}^{b_i} (c_2^{(i)}(x))^{-1/2} dx, \quad M_1 \neq 0, \quad M_{\sigma} \neq 0.$$

By Lemma 1 of⁽¹⁾, all zeros of the function $H(z)$ are located in the z -plane in a certain strip D_h , bounded by the straight lines $\operatorname{Re}(z) = \pm h$ ($h > 0$).

With the aid of the concrete asymptotics of the solution of the spectral problem (4)–(5) and of the existence and well-posedness theorems for the solutions of problems A and B (Theorems 24, 25, 28 of⁽¹⁾), we prove the following theorems.

Theorem 1. *Under conditions a), b) and A, problem B has a unique solution $w^{(i)}(x, t)$, depending continuously on the right-hand sides of equations (1) and representable for all $x \in [a_i, b_i]$, $t \in (0, T]$ and $x \in (a_i, b_i)$, $t \in (0, T]$ by the formula*

$$\begin{aligned} w^{(i)}(x, t) = & -\frac{1}{2\pi\sqrt{-1}} \lim_{\omega \rightarrow \infty} \int_{\eta-\omega\sqrt{-1}}^{\eta+\omega\sqrt{-1}} e^{\lambda t} d\lambda \sum_{j=1}^n \int_{a_j}^{b_j} G^{(i,j)}(x, \xi, \lambda) \times \\ & \times \int_0^t e^{-\lambda \tau} f^{(j)}(\xi, \tau) (c_2^{(j)}(\xi))^{-1} d\tau d\xi, \quad \eta > h^2 \quad (i = 1, \dots, n). \end{aligned} \quad (6)$$

All derivatives of $w^{(i)}(x, t)$ entering into problem B are obtained by differentiating under the sign of the contour integral in (6).

Let us call $A_{(0)}$, $A_{(1)}$, $A_{(2)}$ the problems A with boundary conditions (2), respectively: 1) containing no derivatives with respect to t ; 2) containing the derivative $\partial u^{(i)}/\partial t$ for at least one value of i and s ; 3) containing the derivative $\partial^2 u^{(i)}/\partial t \partial x$ for at least one value of i and s .

Theorem 2. Problem $A_{(k)}$ has, under conditions a), A , $B_{(k)}$, a unique solution, depending continuously on the right-hand sides of the boundary conditions (2) and representable for all $x \in [a_i, b_i]$, $t > 0$ and $x \in (a_i, b_i)$, $t \geq 0$ by the formula

$$v^{(i)}(x, t) = \frac{1}{2\pi\sqrt{-1}} \lim_{\omega \rightarrow \infty} \int_{\eta - \sqrt{-1}\omega}^{\eta + \sqrt{-1}\omega} e^{\lambda t} \frac{\Delta^{(i)}(x, \lambda)}{\lambda \Delta(\lambda)} d\lambda, \quad \eta > h^2 \quad (i = 1, \dots, n) \quad (7)$$

All derivatives of the solution $v^{(i)}(x, t)$ entering into problem $A_{(k)}$ are obtained by differentiating under the sign of the contour integral in (7).

2. Usually, when solving problem (1)–(3) on the half-line $t > 0$ by the operational method (with the aid of the one-sided Laplace transform with respect to the variable t), one assumes that the right-hand sides of equations (1) are Laplace-transformable with respect to t and makes the following assumptions: 1) the solution of the problem and all derivatives of the solution entering into the formulation of the problem are Laplace-transformable; 2) the operations of differentiation $\partial/\partial x$, $\partial^2/\partial x^2$ commute with L_t (L_t is the Laplace integral); 3) the operations of passage to the limit as $x \rightarrow a_i \pm 0$, $x \rightarrow b_i \pm 0$ commute with L_t . Naturally, by this route only a formal solution can be obtained. Thus, for example, problem (1)–(3) is solved in (6) with coefficients $c_l^{(i)}(x)$ ($l = 0, 1, 2$; $n = 2$), independent of x , and with boundary conditions containing no derivatives with respect to t .

Let $S\{a, b\}$ be the set of all functions $\Phi(x, t)$, defined for $x \in [a, b]$ (or $x \in (a, b)$), $0 \leq t < \infty$, integrable with respect to t on every finite interval $[0, T]$ for any $x \in [a, b]$ (respectively, $x \in (a, b)$), and for which the Laplace integral L_t converges at some point $\lambda_0 = \lambda_0(\Phi)$ uniformly with respect to $x \in [a, b]$ (respectively, $x \in (a, b)$). To each fixed function $\Phi(x, t) \in S\{a, b\}$ there corresponds the set $E\{\Phi, a, b\}$ of all real values λ for which L_t converges uniformly with respect to x . The lower bound of the set $E\{\Phi, a, b\}$ will be called the **abscissa of convergence** of the integral $L_t\{\Phi(x, t)\}$.

Theorem 3. Under conditions a), A , $B_{(k)}$, the solution $v^{(i)}(x, t)$ of problem $A_{(k)}$, together with all derivatives entering this problem, belongs to the set $S\{a_i, b_i\}$. The abscissa of convergence of their Laplace integrals does not exceed h^2 .

Theorem 4. Let the right-hand sides $f^{(i)}(x, t)$ of equations (1) be continuous for $x \in [a_i, b_i]$, $t > 0$, and let, on the interval $[a_i, b_i]$, all the functions

$$\frac{\partial^{s+m}}{\partial t^s \partial x^m} f^{(i)}(x, t)$$

($m = 0, 1, 2$; $s = 0, 1$) belong to the set $S\{a_i, b_i\}$ with abscissa of convergence $\mu^{(i)} \geq 0$ of their Laplace integrals.

Then, under conditions a), A, the solution $w^{(i)}(x, t)$ of problem B, together with all derivatives entering this problem, belongs to the set $S\{a_i, b_i\}$. The abscissa of convergence of their Laplace integrals does not exceed $\max(\mu^{(i)}, h^2)$.

It is clear that, under the conditions of Theorems 3 and 4, assumptions 1), 2), 3) are certainly satisfied for problem (1)–(3), which makes it possible to solve this problem directly by the operational method.

As a result, for the solution of the problem,

$$u^{(i)}(x, t) = \frac{1}{2\pi\sqrt{-1}} \lim_{\omega \rightarrow \infty} \int_{\eta - \omega\sqrt{-1}}^{\eta + \omega\sqrt{-1}} e^{\lambda t} \times \\ \times \left(\frac{\Delta^{(i)}(x, \lambda)}{\lambda \Delta(\lambda)} - \sum_{j=1}^n \int_{a_j}^{b_j} G^{(i,j)}(x, \xi, \lambda) \bar{f}^{(j)}(\xi, \lambda) (c_2^{(j)}(\xi))^{-1} d\xi \right) d\lambda, \quad (8) \\ \eta > \max(\mu^{(i)}, h^2) \quad (i = 1, \dots, n),$$

which is valid for all $x \in [a_i, b_i]$, $t > 0$, and $x \in (a_i, b_i)$, $t \geq 0$.

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Note: Figure translations are in progress. See original paper for figures.

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