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**Abstract**

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**MATHEMATICS**

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# **CONTINUATION AND A FUNCTIONAL EQUATION FOR ZETA-FUNCTIONS WITH NON-ABELIAN CHARACTERS OF SIMPLE ALGEBRAS OVER NUMBER FIELDS**

*(Presented by Academician I. M. Vinogradov on 29 XI 1967)*

1. Zeta-functions of algebras over number fields were first considered by H. Hey (<sup>1,2</sup>). He defined zeta-functions of algebras with division and proved their analytic continuation and a functional equation. Eichler (<sup>3</sup>) generalized these results to zeta-functions with abelian (i.e., one-dimensional) characters. Fujisaki (<sup>4,5</sup>) considered zeta-functions with abelian characters for simple algebras and, reducing them to zeta-functions of the corresponding algebras with division, proved for them continuation and a functional equation.

With the appearance of the theory of Hecke operators (<sup>6</sup>), the need arose to study zeta-functions connected with representations of the ring of Hecke operators on spaces of automorphic functions. This formulation of the question leads to the problem of studying zeta-functions with non-abelian characters, where the non-abelian characters that occur are positive-definite zonal spherical functions corresponding to automorphic functions. In general form this problem was posed in lectures of Godement (<sup>7</sup>), and there, by the Iwasawa-Tate method, was solved for algebras with division. Tamagawa (<sup>8,9</sup>) developed the theory of the Euler product for non-abelian zeta-functions. Kinoshita (<sup>10</sup>) proved continuation and a functional equation for non-abelian zeta-functions of the full matrix algebra over the field of rational numbers.

We have obtained a general result in this direction: analytic continuation and a functional equation have been proved for zeta-functions with non-abelian characters of an arbitrary simple central algebra over a field of algebraic numbers. We proceed to a detailed formulation of the results.

2. Let  $A$  be a simple central algebra over a finite field of algebraic numbers  $k$ ;  $G$  the multiplicative group of  $A$ ;  $\mathfrak{O}$  a maximal order in  $A$ . For each valuation  $\mathfrak{p}$  of the field  $k$ , denote by  $k_{\mathfrak{p}}$  the completion of  $k$  with respect to  $\mathfrak{p}$ ;  $A_{\mathfrak{p}} = A \otimes_k k_{\mathfrak{p}}$ ;  $G_{\mathfrak{p}}$  the multiplicative group of  $A_{\mathfrak{p}}$ . For non-Archimedean

$\mathfrak{p}$ , denote by  $\mathfrak{D}_{\mathfrak{p}}$  the closure of  $\mathfrak{D}$  in  $A_{\mathfrak{p}}$ , and by  $U_{\mathfrak{p}}$  the subgroup generated by those  $u_{\mathfrak{p}} \in G_{\mathfrak{p}}$  for which  $u_{\mathfrak{p}}\mathfrak{D}_{\mathfrak{p}} = \mathfrak{D}_{\mathfrak{p}}$ . For Archimedean  $\mathfrak{p}$  put  $U_{\mathfrak{p}} = \{u_{\mathfrak{p}} \in G_{\mathfrak{p}}, uu_{\mathfrak{p}}^* = 1\}$ , where  $a_{\mathfrak{p}} \mapsto a_{\mathfrak{p}}^*$  is a positive involution of the algebra  $A_{\mathfrak{p}}$ . The subgroup  $U_{\mathfrak{p}} \subset G_{\mathfrak{p}}$  is compact for all  $\mathfrak{p}$ ; for non-Archimedean  $\mathfrak{p}$  it is, in addition, open. For  $x_{\mathfrak{p}} \in A_{\mathfrak{p}}$  put

$$V_{\mathfrak{p}}(x_{\mathfrak{p}}) = |Nx_{\mathfrak{p}}|_{\mathfrak{p}},$$

where  $N$  is the norm in the regular representation of  $A_{\mathfrak{p}}$  over  $k_{\mathfrak{p}}$ ;  $|\cdot|_{\mathfrak{p}}$  is the usual norm in the complete field  $k_{\mathfrak{p}}$ , for which the product formula <sup>(11)</sup> holds. Put, for  $x_{\mathfrak{p}} \in A_{\mathfrak{p}}$ ,

$$\Phi_{\mathfrak{p}}(x_{\mathfrak{p}}) = \begin{cases} \exp(-\pi \operatorname{Tr}(x_{\mathfrak{p}}x_{\mathfrak{p}}^*)), & \text{if } \mathfrak{p} \text{ is Archimedean,} \\ \text{the characteristic function of the maximal order } \mathfrak{D}_{\mathfrak{p}} \subset A_{\mathfrak{p}}, & \text{if } \mathfrak{p} \text{ is non-Archimedean.} \end{cases}$$

where  $\operatorname{Tr}$  is the reduced trace over the field  $\mathbf{R}$  of real numbers;  $x_{\mathfrak{p}} \mapsto x_{\mathfrak{p}}^*$  is a positive involution of  $A_{\mathfrak{p}}$ .

3. Let  $J$  be the group of ideals of the group  $G$ , i.e. the restricted direct product <sup>(11)</sup> of the groups  $G_{\mathfrak{p}}$  with respect to the subgroups  $U_{\mathfrak{p}}^*$  over all valuations  $\mathfrak{p}$  of the field  $k$ . Denote by  $\Gamma \cong G$  the subgroup of principal ideals and put  $U = \prod_{\mathfrak{p}} U_{\mathfrak{p}}$ . For  $g = (\dots g_{\mathfrak{p}} \dots) \in J$  put

$$V(g) = \prod_{\mathfrak{p}} V(g_{\mathfrak{p}}); \quad \Phi(g) = \prod_{\mathfrak{p}} \Phi_{\mathfrak{p}}(g_{\mathfrak{p}}).$$

4. Let  $H$  be a locally compact group;  $V \subset H$  a compact subgroup. Denote by  $L(H, V)$  the algebra of all complex continuous functions  $f$  on  $H$  with compact support which, for all  $v, v' \in V$ ,  $h \in H$ , satisfy the condition  $f(hv'v) = f(h)$ . Multiplication in  $L(H, V)$  is defined as convolution:

$$(f * \varphi)(x) = \int_H f(xh^{-1})\varphi(h) dh, \quad x \in H.$$

Recall that a continuous function  $\omega$  on  $H$  is called a (zonal) spherical function with respect to the subgroup  $V$  <sup>(8,9,12)</sup> if for any  $h, h' \in H$

$$\int_V \omega(hvh') dv = \omega(h)\omega(h'),$$

where  $dv$  is the Haar measure on  $V$ , normalized by the condition  $\int_V dv = 1$ .

5. Let  $\omega$  be a spherical function on  $J$  with respect to  $U$ ; then

$$\omega = \prod_{\mathfrak{p}} \omega_{\mathfrak{p}},$$

where  $\omega_{\mathfrak{p}}$  are spherical functions on  $G_{\mathfrak{p}}$  with respect to  $U_{\mathfrak{p}}$  <sup>(8)</sup>. For all  $\mathfrak{p}$ ,  $A_{\mathfrak{p}}$  is a simple algebra over  $k_{\mathfrak{p}}$ , so that  $A_{\mathfrak{p}} = M_{r_{\mathfrak{p}}}(D_{\mathfrak{p}})$ , where  $D_{\mathfrak{p}}$  is a division algebra. The set of spherical functions on  $G_{\mathfrak{p}}$  (with respect to  $U_{\mathfrak{p}}$ ) is parametrized by  $r_{\mathfrak{p}}$  complex parameters <sup>(9)</sup>. We describe this parametrization for Archimedean  $\mathfrak{p}$ . Put

$$T_{\mathfrak{p}} = \{t = (t_{ij}) \in G_{\mathfrak{p}}, t_{ij} \in D_{\mathfrak{p}}, t_{ij} = 0 \text{ for } i > j, t_{ii} \in \mathbf{R}, t_{ii} > 0\}.$$

Then  $G_{\mathfrak{p}} = U_{\mathfrak{p}}T_{\mathfrak{p}}$ . Let  $(s(\mathfrak{p})) = (s_1(\mathfrak{p}), \dots, s_{r_{\mathfrak{p}}}(\mathfrak{p}))$  be arbitrary complex numbers. For  $g_{\mathfrak{p}} \in G_{\mathfrak{p}}$ ,  $g_{\mathfrak{p}} = u_{\mathfrak{p}}t_{\mathfrak{p}}$ , where  $u_{\mathfrak{p}} \in U_{\mathfrak{p}}$ ,  $t_{\mathfrak{p}} \in T_{\mathfrak{p}}$ , put

$$\alpha_{(s(\mathfrak{p}))}(g_{\mathfrak{p}}) = \alpha_{(s(\mathfrak{p}))}(t_{\mathfrak{p}}) = \prod_{i=1}^{r_{\mathfrak{p}}} t_{ii}^{\nu(-s_i(\mathfrak{p})+(i-1))},$$

where  $\nu = [D_{\mathfrak{p}} : \mathbf{R}]$ . Then the function

$$\omega_{(s(\mathfrak{p}))}(g_{\mathfrak{p}}) = \int_{U_{\mathfrak{p}}} \alpha_{(s(\mathfrak{p}))}(g_{\mathfrak{p}}^{-1}u_{\mathfrak{p}}) du_{\mathfrak{p}}$$

is a spherical function on  $G_{\mathfrak{p}}$  with respect to  $U_{\mathfrak{p}}$ , and every spherical function on  $G_{\mathfrak{p}}$  is obtained in the indicated way by a suitable choice of the parameters  $s_i(\mathfrak{p})$ . We shall call the parameters  $s_i(\mathfrak{p})$  ( $i = 1, \dots, r_{\mathfrak{p}}$ ) the roots of the spherical function  $\omega_{\mathfrak{p}} = \omega_{(s(\mathfrak{p}))}$  and associate to each spherical function  $\omega_{\mathfrak{p}}$  the polynomial

$$P_{\omega_{\mathfrak{p}}}(z) = \prod_{i=1}^{r_{\mathfrak{p}}} (r_{\mathfrak{p}}z - s_i(\mathfrak{p}))(r_{\mathfrak{p}}z - s_i(\mathfrak{p}) - 1),$$

where  $s_i(\mathfrak{p})$  are the roots of  $\omega_{\mathfrak{p}}$ .

6. A continuous function  $f$  on  $J$  is called  $\Gamma$ -automorphic if:

- I.  $f(ug\gamma) = f(g)$  for all  $u \in U$ ,  $g \in J$ ,  $\gamma \in \Gamma$ .
- II. For every function  $\varphi \in L(J, U)$  there exists a complex number  $\lambda_{\varphi}$  such that  $\varphi * f = \lambda_{\varphi} \cdot f$ .

To every nonzero  $\Gamma$ -automorphic function  $f$  on  $J$  there corresponds uniquely a spherical function  $\omega$  on  $J$  (with respect to  $U$ ), which for all  $g, g' \in J$  satisfies the condition

$$\int_U f(gug') du = \omega(g)f(g'). \quad (8)$$

We shall say that  $\omega$  belongs to  $f$ . By the spectrum  $s(\Gamma)$  of the discrete subgroup  $\Gamma \subset J$  we shall mean the set of all spherical functions  $\omega$  on  $J$  with respect to  $U$  that belong to some nonzero  $\Gamma$ -automorphic functions on  $J$ , are positive-definite functions on  $J$ , and for every  $\xi$  in the center of  $J$  and  $g \in J$  satisfy the relation  $\omega(\xi g) = \omega(g)$ .

7. Let  $\Gamma$  be the group of principal ideles,  $\omega \in s(\Gamma)$ . The zeta-function of the algebra  $A$  with character  $\omega$  is the function

$$\begin{aligned}\zeta(z, \omega) &= \int_J \Phi(g) \omega(g^{-1}) V(g)^z dg = \\ &= \prod_{\mathfrak{p}} \int_{G_{\mathfrak{p}}} \Phi_{\mathfrak{p}}(g_{\mathfrak{p}}) \omega_{\mathfrak{p}}(g_{\mathfrak{p}}^{-1}) V_{\mathfrak{p}}(g_{\mathfrak{p}})^z dg_{\mathfrak{p}} = \prod_{\mathfrak{p}} \zeta_{\mathfrak{p}}(z, \omega_{\mathfrak{p}}),\end{aligned}$$

where  $dg = \prod_{\mathfrak{p}} dg_{\mathfrak{p}}$  is Haar measure on  $J$ , and the local measures  $dg_{\mathfrak{p}}$  on  $G_{\mathfrak{p}}$ , for non-Archimedean  $\mathfrak{p}$ , are normalized by the condition

$$\int_{U_{\mathfrak{p}}} dg_{\mathfrak{p}} = 1.$$

The function  $\zeta(z, \omega)$  is regular in the domain  $\operatorname{Re} z > 1$ . The  $\mathfrak{p}$ -factors  $\zeta_{\mathfrak{p}}(z, \omega_{\mathfrak{p}})$  have been computed explicitly by Tamagawa (9); for Archimedean  $\mathfrak{p}$  they are expressed in terms of the gamma-function, while for non-Archimedean  $\mathfrak{p}$  they are rational functions of  $p^{-z}$ , where  $p$  is the prime number divisible by  $\mathfrak{p}$ . Our main result is formulated as follows:

**Main theorem.** *In the notation and assumptions introduced above, the following assertions hold:*

- I. The function  $\zeta(z, \omega)$  admits a meromorphic continuation to the whole  $z$ -plane.  
II. The function

$$\zeta(z, \omega) \prod_{\text{arch}} P_{\omega_{\mathfrak{p}}}(z)$$

is entire.\*

- III. The function  $\zeta(z, \omega)$  satisfies a functional equation of the form

$$\zeta(z, \omega) = W(\omega) \Delta^{1/2-z} \zeta(1-z, \bar{\omega}),$$

where  $W(\omega)$  is a constant depending only on  $\omega$ ;  $|W(\omega)| = 1$ ;  $\Delta$  is the absolute discriminant of the algebra  $A$ .

The proof is based on combining the analysis of weight functions on the Archimedean components with the Poisson formula for the additive group of adeles of the algebra  $A$ .

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## REFERENCES

1. K. Hey, *Analytische Zahlentheorie in System hyperkomplexer Zahlen*, Diss., Hamburg, 1929.
2. M. Deuring, *Algebren*, N. Y., 1948.
3. M. Eichler, J. reine u. angew. Math., **179**, 227 (1938).
4. G. Fujisaki, J. Fac. Sci. Univ. Tokyo, sect. I, **7**, 567 (1958).
5. G. Fujisaki, J. Fac. Sci. Univ. Tokyo, sect. I, **9**, 293 (1962).
6. E. Hecke, Math. Ann., **114**, 1 (1937).
7. R. Godement, Sem. Bourbaki 1958/1959, Exp. 171, 176.
8. T. Tamagawa, J. Fac. Sci. Univ. Tokyo, sect. I, **8**, 363 (1960).
9. T. Tamagawa, Ann. Math., **77**, 387 (1963).
10. M. Kinoshita, J. Math. Soc. Japan, **17**, No. 4, 374 (1965).
11. J. Tate, *Fourier Analysis in Number Fields and Hecke's Zeta-functions*, Thesis, Princeton Univ., 1950.
12. Harish-Chandra, Am. J. Math., **80**, 241, 553 (1958).

\* This assertion remains valid if, instead of the product over all Archimedean  $\mathfrak{p}$ , one takes an arbitrary one of these polynomials  $P_{\omega}(z)$ .

*Note: Figure translations are in progress. See original paper for figures.*

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