

SMOOTH KNOTS IN A MANIFOLD OF THE HOMOTOPY TYPE OF A SPHERE

S. G. Smirnov

1968

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196801.18240>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.835

MATHEMATICS

S. G. SMIRNOV

SMOOTH KNOTS IN A MANIFOLD OF THE HOMOTOPY TYPE OF A SPHERE

(Presented by Academician P. S. Aleksandrov on 19 IV 1968)

Let M^n be a smooth compact manifold. Denote by $\text{Iso}(S^m, M^n)$ the set of smooth-isotopy classes of embeddings of the m -sphere S^m in M^n , and by $\text{Iso}_p(S^m, M^n)$ the corresponding set in the combinatorial category. Haefliger ⁽¹⁾ and Zeeman ⁽²⁾ computed these sets in the case $M^n = S^k \times D^{n-k}$, under the stability conditions for the group $\pi_m(S^k)$ and $k > 2$, $n - k > 2$, $m > 3$, $n - m > 2$. In the present note, under the same restrictions, we first study the set $\text{Iso}^*(S^m, M^n)$ of combinatorial-isotopy classes of smooth embeddings of S^m in M^n , where M^n has the homotopy type of the sphere S^k . We then show that in the homotopically unstable region, for given m, n, k , this set depends on the differential type of M^n .

§ 1. Let $m > 3$, $n - m > 2$. Denote the n -dimensional disk by D^n and $\text{Iso}(S^m, D^n)$ by C_m^{n-m} . It is known (see ⁽¹⁾) that C_m^{n-m} is an abelian group.

Every embedding $i : D^n \subset M^n$ induces a map

$$i_* : C_m^{n-m} \longrightarrow \text{Iso}(S^m, M^n).$$

Elements of $\text{Im } i_* \subset \text{Iso}(S^m, M^n)$ will be called local knots. Let M^n be simply connected. We shall define the action of $\text{Im } i_*$ on $\text{Iso}(S^m, M^n)$.

Let $\varphi \in \text{Iso}(S^m, M^n)$, $\alpha \in \text{Im } i_*$, and let $f_\varphi : S^m \subset M^n$, $f_\alpha : S^m \subset D_0^n \subset M^n$ be such representatives of φ and α that $\text{Im } f_\varphi \cap D_0^n = \emptyset$. Join in M^n points $x \in \text{Im } f_\varphi$, $y \in \text{Im } f_\alpha$ by a smooth path not intersecting $\text{Im } f_\varphi$, $\text{Im } f_\alpha$, and take the connected sum $\text{Im } f_\varphi \# \text{Im } f_\alpha$ along a tube lying in an ε -neighborhood of this path. Since M^n is simply connected, $m > 3$, $n - m > 2$, it is clear that the operation described is possible for any φ, α , and the smooth-isotopy class of the resulting embedding $f_{\varphi+\alpha}$ depends only on φ and α . Let t be the coordinate on the interval I . We shall call a map $F_t : X \times I \rightarrow Y \times I$ fiberwise with respect to t if it commutes with the projection of the direct product onto the factor I .

Lemma 1. *Two elements of the set $\text{Iso}(S^m, M^n)$ can be transformed one into the other by the group $\text{Im } i_*$ if and only if their representatives are combinatorially isotopic.*

Proof. a) Let φ, α be as above, $f_\varphi, f_{\varphi+\alpha} : S^m \subset M^n$. Then there exists a disk $D^n \subset M^n$ outside which $\text{Im } f_\varphi$ and $\text{Im } f_{\varphi+\alpha}$ coincide, and $D^m \cap \text{Im } f_\varphi$ and $D^n \cap \text{Im } f_{\varphi+\alpha}$ are disks D_1^m, D_2^m in D^n , coinciding on the boundary:

$$\partial D_1^m = \partial D_2^m \subset \partial D^n.$$

By Theorem 1 of ⁽³⁾, D_1^m and D_2^m are combinatorially isotopic in D^n relative to the boundary. This isotopy extends to an isotopy

$$f_\varphi \rightarrow f_{\varphi+\alpha},$$

identical on the complement of the disk D_1^m .

- b) Let $f, g : S^m \subset M^n$ be smooth embeddings; $D_1^m \subset S^m$, $D_2^m = S^m \setminus D_1^m$, and $f_t : f \rightarrow g$ a combinatorial isotopy. By Hirsch' s theorem on smoothing a contractible combinatorial submanifold ⁽⁴⁾, Theorem B), f_t may be assumed smooth on $S^m \setminus \{*\}$, where $*$ $\in D_2^m$. By Cerf' s theorem on the isotopy of D^n in M^n , by a smooth-isotopic replacement of g we may arrange that g and f coincide on D_1^m ; in view of the simple connectedness of M^n , f_t may be regarded as fixed on D_1^m . Then the restriction of f_t to D_2^m determines a fiber-

with respect to t a combinatorial embedding $F_t : D_2^m \times I \subset M^n \times I$, smooth for $t = 0, 1$ and constant in t on $\partial D_2^m \times I$. According to Theorem B of [4], in $M^n \times I$ there exists a smooth closed neighborhood of the image $\text{Im } F_t$, which is a disk $D_2^n \times I$, fiberwise embedded in $M^n \times I$ with respect to t , and moreover

$$\text{Im } F_t \cap (\partial D_2^n \times I) = F_t(\partial D_2^m \times I).$$

Let $H_t : D_2^n \times I \rightarrow D^n \times I$ be a diffeomorphism, fiberwise in t , onto the standard disk such that $H_t \cdot F_t(\partial D_2^m \times I) \subset \partial D^n \times I$ is the product over I of the standard embedding S^{m-1} in S^{n-1} . Then the embeddings

$$H_0 \cdot F_0, \quad H_1 \cdot F_1 : (D_2^m, \partial D_2^m) \subset (D^n, \partial D^n)$$

represent elements $\alpha, \beta \in C_m^{n-m}$, and the image $\text{Im } H_t \cdot F_t$ determines a combinatorial isotopy $\alpha \rightarrow \beta$. Add, as above, in $D^n \times (0)$ to $\text{Im } H_0 \cdot F_0$ a local embedding $f_{\beta-\alpha} : S^m \subset M^n$. The new embedding

$$(H_0 \cdot F_0 \# f_{\beta-\alpha}) : D^m \subset D^n$$

is smoothly isotopic rel boundary to the embedding $H_1 \cdot F_1$. Therefore the embedding $\tilde{f} : S^m \subset M^n$, equal to f on D_1^m and to

$$H_0^{-1}(H_0 \cdot F_0 \# f_{\beta-\alpha})$$

on D_2^m , differs from f by a local knot and is smoothly isotopic to g in M^n . Thus,

$$\text{Iso}(S^m, M^n) / \text{Im } i_* = \text{Iso}^*(S^m, M^n).$$

§ 2. **Lemma 2.** *Let M^{n+1} be a smooth compact manifold with boundary, having the homotopy type of the sphere S^{k+1} ; let $\pi_1(\partial M^{n+1}) = 0$ and $k > 2$, $n - k > 2$. Then*

$$M^{n+1} = D_1^{n+1} \cup D_2^{n+1},$$

where

$$D_1^{n+1} \cap D_2^{n+1} = \partial D_1^{n+1} \cap \partial D_2^{n+1} = S^k \times D^{n-k},$$

and the embedding $j : S^k \times D^{n-k} \subset \partial D_1^{n+1}$ is canonical.

This lemma follows directly from Theorem 6.5 of [5].

We shall call a smooth embedding $f : S^{m+1} \subset M^{n+1}$ ($m > 2$, $n - m > 2$) a suspension if

$$\text{Im } f \cap (S^k \times D^{n-k}) = S^m.$$

It follows easily from Theorem 1 of [3] that suspensions over combinatorially isotopic embeddings of S^m in $S^k \times D^{n-k}$ are combinatorially isotopic in M^{n+1} .

It is known (see [1, 2]) that for $k, m, n - k, n - m > 2$, $\text{Iso}(S^m, S^k \times D^{n-k})$ is an abelian group with respect to connected sum;

$$\text{Iso}_p(S^m, S^k \times D^{n-k}) = \text{Iso}^*(S^m, S^k \times D^{n-k})$$

and

$$\text{Iso}(S^m, S^k \times D^{n-k}) = \text{Iso}^*(S^m, S^k \times D^{n-k}) + C_m^{n-m}.$$

The mapping i_* in this case is a monomorphism, and the standard embedding

$$j : S^k \times D^{n-k} \subset S^n$$

induces a projection of $\text{Iso}(S^m, S^k \times D^{n-k})$ onto C_m^{n-m} with kernel $\text{Iso}^*(S^m, S^k \times D^{n-k})$. Therefore every element of $\text{Iso}^*(S^m, S^k \times D^{n-k})$ is realized by a sphere $S^m \subset S^k \times D^{n-k}$ bounding a smoothly embedded disk D_1^{m+1} in D_1^{n+1} , but, possibly, not bounding such a disk in D_2^{n+1} . Thus we have, for every M^{n+1} satisfying Lemma 2 and $m > 2$, $n - m > 2$, a partial mapping

$$\Sigma : \text{Iso}^*(S^m, S^k \times D^{n-k}) \rightarrow \text{Iso}^*(S^{m+1}, M^{n+1}).$$

It is easy to see that the domain of definition of Σ is a subgroup $G_m(M^{n+1}) \subset \text{Iso}^*(S^m, S^k \times D^{n-k})$. It coincides with the group $\text{Iso}^*(S^m, S^k \times D^{n-k})$ if

$$2n > 3(m+1)$$

(and hence $C_m^{n-m} = 0$) or if $M^{n+1} = S^{k+1} \times D^{n-k}$; in the latter case Σ is a group homomorphism.

§ 3. **Theorem.** *Let M^{n+1} satisfy Lemma 2 and let $m > k > 3$, $n - m > 2$. Then the mapping*

$$\Sigma : g_m(M^{n+1}) \rightarrow \text{Iso}^*(S^{m+1}, M^{n+1})$$

is an epimorphism for $m < 2k$ and an isomorphism for $m < 2k - 1$.

Proof. a) Let $f : S^{m+1} \subset M^{n+1}$ be a smooth embedding. It is easy to see that there exist smooth disks

$$(D_i^{n-k}, \partial D_i^{n-k}) \subset (M^{n+1}, \partial M^{n+1}), \quad i = 1, 2,$$

such that $D_i^{n-k} \subset D_i^{n+1}$,

$$D_i^{n-k} \cap S^k \times D^{n-k} = \emptyset$$

and $M^{n+1} \setminus D_i^{n-k}$ is contractible. Denote

$$\text{Im } f \cap D_i^{n-k}$$

by M_i^{m-k} , and

$$M^{n+1} \setminus D_2^{n-k}$$

by E^{n+1} ,

$$\text{Im } f \setminus D_2^{n-k} = \text{Im } f \setminus M_2^{m-k}$$

by V^{m+1} . Let

$$U^{n+1} = D_1^{n-k} \times D_\varepsilon^k$$

be a closed tubular neighborhood of D_1^{n-k} in M^{n+1} . Denote

$$\text{Im } f \cap U^{n+1}$$

by M_1^{m+1} ; then M_1^{m+1} contracts to M_1^{m-k} . Since $m > 4$, $m < 2k$, the embedding M_1^{m-k} in V^{m+1} is homotopic to zero, and

$$\pi_l(\partial M^{m+1}) = \pi_l(V^{m+1})$$

for $l \leq m - k$.

Inductively in l , one may assume M_1^{m+1} to be $(l-1)$ -connected ($0 \leq l \leq m-k$). Let $\lambda \in \pi_l(M_1^{m+1})$ be one of the generators. We realize λ by a sphere $S^l \subset \partial M_1^{m+1}$, and span in

$$V^{m+1} \setminus \text{Int } U^{n+1}$$

over S^l an embedded disk D^{l+1} —this is possible in view of the condition $m < 2k$. The relative $(l+1)$ -spheroid $(D^{l+1},$

$S^l) \subset (E^{n+1}, U^{n+1})$ is null-homotopic, since $E^{n+1} \sim U^{n+1} \sim *$. Extend (D^{l+1}, S^l) to an embedding in $E^{n+1} \setminus \text{Int } U^{n+1}$ of such a disk D^{l+2} that

$$\partial D^{l+2} = D^{l+1} \cup D_1^{l+1},$$

where $D_1^{l+1} = D^{l+2} \cap U^{n+1}$. Now attach to U^{n+1} a closed tubular neighborhood of the disk D^{l+2} in E^{n+1} . As a result we again obtain a disk; denote it by $\widetilde{U}^{n+1}$, and denote $\text{Im } f \cap \widetilde{U}^{n+1}$ by $\widetilde{M}^{m+1}$. Now, if $\text{Int } D^{l+2} \cap \text{Im } f = \emptyset$, then $\widetilde{M}^{m+1}$

is obtained from M^{m+1} by attaching a handle with core disk D^{l+1} , which kills the generator $\lambda \in \pi_l(M^{m+1})$. The boundary $\partial \widetilde{M}^{m+1}$ of the manifold $\widetilde{M}^{m+1}$ is obtained from ∂M^{m+1} by a Morse modification killing the same generator λ in the group $\pi_l(\partial M^{m+1}) = \pi_l(M^{m+1})$. The condition $\text{Int } D^{l+2} \cap \text{Im } f = \emptyset$ is automatically satisfied for $l = 0$, since we have $n - m > 2$.

Let $l > 0$ and let

$$\Pi_1 = (D^{l+2} \setminus D^{l+1}) \cap \text{Im } f$$

be a manifold with boundary

$$\partial \Pi_1 = \text{Int } D_1^{l+1} \cap \text{Im } f.$$

Then, in constructing $\widetilde{U}^{n+1}$, in addition to the handle P_λ^{l+1} , to M^{m+1} there is attached, along the embedding $\partial \Pi_1 \subset \partial M^{m+1}$, the pair $(\widetilde{\Pi}_1, \partial \widetilde{\Pi}_1)$ —a closed tubular neighborhood $(\Pi_1, \partial \Pi_1)$ in $V^{m+1} \setminus \text{Int } U^{n+1}$. The dimension of the manifold $\partial \Pi_1$, equal to $l + m - n + 1$, is less than the connectivity of the manifold M^{m+1} ; moreover, from the condition $m < 2k$ it follows that $2 \dim \partial \Pi_1 < m$. Therefore $\partial \widetilde{\Pi}_1 \subset \partial M^{m+1}$ lies inside some disk $D_1^m \subset \partial M^{m+1}$. Consequently,

$$M^{m+1} \cup_{\partial} \Pi_1 \approx M^{m+1} \setminus (\Pi_1 / \partial \Pi_1).$$

Now, inductively in i ($0 \leq i \leq \dim \Pi_1$), we shall kill the groups

$$\pi_i(\widetilde{M}^{m+1}) = \pi_i(\Pi_1 / \partial \Pi_1)$$

as above. If in doing this no “second-order intersections”

$$\Pi_2 = (D^{l+2} \setminus D^{l+1}) \cap \text{Im } f$$

arise, then as a result $\widetilde{M}^{m+1}$ will become $(l-1)$ -connected. The group $\pi_l(\widetilde{M}^{m+1})$ is then obtained from the group $\pi_l(M^{m+1})$ by imposing the relation $\lambda = 0$. If, at some step of killing the homotopy of $\widetilde{\Pi}_1$, we have $\Pi_2 \neq \emptyset$, then we begin to kill the homotopy of its tubular neighborhood $\widetilde{\Pi}_2$ in the same way as was done with $\widetilde{\Pi}_1$. In doing so a “third-order intersection” Π_3 may arise.

In the general case there arises a chain of successive intersections:

$$\Pi_1, \Pi_2, \Pi_3, \dots$$

It is finite, since from the condition $n - m > 2$ it follows that

$$\dim \Pi_{s+1} < \dim \Pi_s.$$

Therefore the killing of the group $\pi_i(M^{m+1})$ is carried out in a finite number of steps. After the groups $\pi_j(M^{m+1})$, $j \leq m - k$, have been killed, the manifold $\widetilde{M}^{m+1}$ becomes contractible, and $\partial \widetilde{M}^{m+1}$ becomes $(m - k)$ -connected. From the condition $m > k > 3$ and Smale's theorem on a pair ((5), Theorem 5.1) it follows that $\widetilde{M}^{m+1} = D^{m+1}$.

Now $\widetilde{U}^{n+1} = D^{n+1}$ intersects $\text{Im } f$ in the disk D^{m+1} and coincides with the original U^{n+1} on ∂M^{n+1} . By Cerf's theorem on the isotopy of D^n in M^n , there exists a diffeotopy Φ_t of the manifold M^{n+1} onto itself, carrying $\widetilde{U}^{n+1}$ to D_1^{n+1} . The restriction Φ_t to $\text{Im } f$ carries f into a concordant embedding.

- b) Let $f_{0,1} = \Sigma g_{0,1}$ be representatives of the element $\varphi \in \text{Iso}^*(S^{m+1}, M^{n+1})$. According to Lemma 1, the embeddings $f_{0,1}$ can be chosen smoothly isotopic. Let

$$f_t : S^{m+1} \times I \subset M^{n+1} \times I$$

be a levelwise in t smooth embedding determined by the isotopy $f_0 \rightarrow f_1$. By a smooth isotopy of $\text{Im } f_t$ in $M^{n+1} \times I$ relative to the "boundary" ($t = 0, 1$), we arrange, as above, that the condition

$$(S^k \times D^{n-k} \times I) \cap \text{Im } f_t = S^m \times I$$

be satisfied. In doing so, the embedding

$$\tilde{g}_t : S^m \times I \subset S^k \times D^{n-k} \times I$$

will not be levelwise in t for $0 < t < 1$. But, by Smale's theorem on the extension of a diffeomorphism ((5), Theorem 3.2) and Cerf's theorem on pseudoisotopy ((6)), $\tilde{g}_t$ is isotopic in $S^k \times D^{n-k} \times I$ relative to the "boundary" ($t = 0, 1$) to an embedding g_t that is levelwise in t . This gives us an isotopy

$$g_0 \rightarrow g_1.$$

From this theorem follows, in particular, Zeeman's result:

$$\text{Iso}^*(S^m, S^k \times D^{n-k}) = \text{Iso}^*(S^{m+1}, S^{k+1} \times D^{n-k})$$

for $m > k > 2$, $n - m > 2$, $m \leq 2k - 1$ (see (2)).

§ 4. We shall show by an example that, in the case when the group $\pi_m(S^k)$ is unstable, the set $\text{Iso}^*(S^m, M^n)$, where $M^n \sim S^k$, depends on the differential type of M^n .

Let χ be the Hopf fibration over S^4 with fiber D^4 ; $\eta = \chi \oplus 3$. Denote by M^8, M^{11} the total spaces of the fibrations χ and η ; then M^8 and M^{11} satisfy Lemma 2. The smooth embedding $f : S^7 = \partial M^8 \subset M^8 \subset M^{11}$ represents a generator β of the free part of the group $\pi_7(S^4) = Z + Z_{12}$. We shall prove that β is not realized even by a combinatorial embedding of S^7 in $S^4 \times D^7$. Indeed, the suspension $\Sigma : \pi_7(S^4) \rightarrow \pi_8(S^5)$ is an epimorphism, under which β goes to the generator $\Sigma\beta \in \pi_8(S^5) = Z_{24}$, and if β were realized by a combinatorial embedding of S^7 in $S^4 \times D^7$, then $\Sigma\beta$ would be realized by a combinatorial embedding of S^8 in $S^5 \times D^7$, i.e. the map

$$\pi : \text{Iso}_P(S^8, S^5 \times D^7) \rightarrow \pi_8(S^5)$$

would be an epimorphism. But, according to (2), $\text{Iso}_P(S^8, S^5 \times D^7) = \pi_6(S^3) = Z_{12}$, and such an epimorphism is impossible. Moreover, the embedding $f : S^7 \subset M^{11}$ is not homotopic and, consequently, not isotopic to a suspended one.

Thus, there is no natural, i.e. projection-commuting into the group $\pi_7(S^4)$, isomorphism between the sets $\text{Iso}^*(S^7, S^4 \times D^7)$ and $\text{Iso}^*(S^7, M^{11})$, and the suspension $\Sigma : g_6(M^{11}) \rightarrow \text{Iso}^*(S^7, M^{11})$ is not an epimorphism.

Analogous examples can be obtained from the Hopf fibration over S^8 with fiber D^8 .

Received
23 II 1968

REFERENCES

1. A. Haefliger, Comm. Math. Helv., 41, 1, 51 (1966).
2. E. C. Zeeman, Seminar on Combinatorial Topology, No. 8, Coventry, 1966.
3. E. C. Zeeman, Ann. Math., 78, 3, 501 (1963).
4. M. W. Hirsch, Comm. Math. Helv., 36, 2, 103 (1961).
5. S. Smale, Matematika, 8, 4, 95 (1964).
6. J. Cerf, Abstracts of Reports of the International Congress of Mathematicians, Moscow, 1966, p. 41.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.