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Abstract

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MATHEMATICS

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ON GÅRDING' S INEQUALITY FOR DEGENERATE ELLIPTIC OPERATORS

(Presented by Academician S. L. Sobolev on 29 XI 1967)

Let Ω be a bounded domain of the $(n + 1)$ -dimensional Euclidean space of points $x = (x_1, \dots, x_n, x_{n+1})$. Let m be a fixed natural number. Suppose that for every $z \in \Omega$ there is given a real homogeneous form $a(z, \xi)$ of the real variables $\xi_1, \dots, \xi_n, \xi_{n+1}$ of degree $2m$, with: 1) the coefficients of the form sufficiently smooth and uniformly continuous; 2) $\inf_{|\xi|=1} a(z, \xi)$ having a positive lower bound in Ω . With every such form one can associate the Dirichlet form

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(z) D^\alpha u D^\beta u, \quad (1)$$

where the coefficients $a_{\alpha\beta}(z)$ are real, symmetric, satisfy condition 1), and, moreover,

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(z) \delta^\alpha \delta^\beta = a(z, \xi). \quad (2)$$

As is known, such forms exist. If $m = 1$, then there exists only one Dirichlet form, and this form is positive definite. For $m > 1$ it may happen that no Dirichlet form is positive definite. From some Dirichlet form belonging to the form $a(z, \xi)$ we construct the Dirichlet integral

$$I(u, u) = \int_{\Omega} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(z) D^\alpha u D^\beta u \, dz. \quad (3)$$

The following assertion is valid.

Gårding' s inequality ⁽¹⁾. There exist positive constants C_1 and C_2 such that

$$I(u, u) \geq C_1 \|u\|_{W_2^m}^2 - C_2 \|u\|_{L_2}^2. \quad (4)$$

This inequality, as is known, plays an important role in the use of functional methods in the theory of partial differential equations.

In the present note we indicate a class of degenerate elliptic operators for which Gårding's inequality ⁽¹⁾ retains its validity. Let E_{n+1}^+ denote the half-space $y > 0$ ($x_{n+1} = y$) of the Euclidean $(n+1)$ -dimensional space of points $z = (x, y)$ ($x = (x_1, \dots, x_n)$). We shall consider a bounded domain Ω^+ lying in the half-space $y > 0$ and adjacent to the hyperplane $y = 0$. Let $C_0^\infty(\Omega^+)$ denote the set of infinitely differentiable functions having compact support contained in Ω^+ . On this set we consider the differential operator

$$\widetilde{D}_y^k u = y^k \frac{\partial^k u}{(y \partial y)^k} = \frac{\partial^k u}{\partial y^k} + \sum_{i=1}^k C_i^{(k)} \frac{\partial^{k-i} u}{y^i \partial y^{k-i}}. \quad (5)$$

For every point $z \in \Omega^+$ we prescribe a real homogeneous form $a(z; \xi, \eta)$ of the real variables $\xi_1, \dots, \xi_n, \eta$ of degree $2m$. In this case

we shall assume that the coefficients satisfy conditions 1) and 2) and, as $y \rightarrow 0$, behave in a definite manner (see below).

To each such form we associate a Dirichlet form of the type

$$\sum_{|\alpha|+k=m} \sum_{|\beta|+l=m} a_{\alpha\beta}^{kl}(z) \widetilde{D}^{\alpha+k} u \widetilde{D}^{\beta+l} u, \quad (6)$$

where $\widetilde{D}^{\alpha+k} = D_x^\alpha \widetilde{D}_y^k$, $\widetilde{D}^{\beta+l} = D_x^\beta \widetilde{D}_y^l$; the coefficients $a_{\alpha\beta}^{kl}$ are real, symmetric, satisfy the conditions indicated above, and possess the property that

$$\sum_{|\alpha|+k=m} \sum_{|\beta|+l=m} a_{\alpha\beta}^{kl}(z) \xi^\alpha \tau^k \xi^\beta \tau^l = a(z; \xi, \tau). \quad (7)$$

Let us take one of these Dirichlet forms and, from it, form an integral of Dirichlet type

$$\tilde{I}_\gamma(u, u) = \int_{\Omega^+} \sum a_{\alpha\beta}^{kl}(z) \widetilde{D}^{\alpha+k} u(z) \widetilde{D}^{\beta+l} u(z) y^{2\gamma} dz. \quad (8)$$

If the coefficients are constant, then in the Fourier–Bessel images the form (8) is written as follows:

$$\tilde{I}_\gamma = C_\gamma^{(m)} \int_{E_{n+1}^+} a(s, t) \left| \tilde{f}_{\gamma-\frac{1}{2}}(s, t) \right|^2 t^{2\gamma} ds dt, \quad (9)$$

where $a(s, t)$ is a homogeneous form of degree $2m$ with constant coefficients, and $\tilde{f}_{\gamma-\frac{1}{2}}$ is the mixed Fourier–Bessel transform (see (4)). To obtain a lower

estimate for the forms just indicated, we introduce the corresponding functional space, considered by the author earlier in papers (2–4). The space $\widetilde{W}_{y,2,\gamma}^k(\Omega^+)$ is defined as the closure of $C_0^\infty(\Omega^+)$ in the norm

$$\|u\|_{\widetilde{W}_{y,2,\gamma}^k}^2 = \int_{\Omega^+} |u|^2 y^{2\gamma} dz + \int_{\Omega^+} |\widetilde{D}_y^k u|^2 y^{2\gamma} dz. \quad (10)$$

We shall characterize differential properties in all the remaining directions as follows. Let l_i be an integer nonnegative number. The functional spaces $W_{x_i,2,\gamma}^{l_i}(\Omega^+)$ ($i = 1, 2, \dots, n$) are defined as the closure of $C_0^\infty(\Omega^+)$ in the usual S. L. Sobolev norm with the weight factor $y^{2\gamma}$. The space $\widetilde{W}_{x,y,2,\gamma}^{l,k}(\Omega^+)$ is defined as the intersection of the corresponding spaces with the norm determined by the formula

$$\sum_{i=1}^n \|u\|_{W_{x_i,2,\gamma}^{l_i}(\Omega^+)}^2 + \|u\|_{\widetilde{W}_{y,2,\gamma}^k(\Omega^+)}^2. \quad (11)$$

This space has many remarkable properties, and for it, in terms of Fourier–Bessel images, there is an equivalent norm (see (4)).

If on $C_0^\infty(\Omega^+)$ we introduce the scalar product

$$(u, v)_j^\gamma = \int_{\Omega^+} \sum_{|\beta|+k=j} \widetilde{D}^{\beta+k} u(z) \widetilde{D}^{\beta+k} v(z) y^{2\gamma} dz \quad (j = 0, 1, 2, \dots, m) \quad (12)$$

and complete the set $C_0^\infty(\Omega^+)$ in the norm

$$\|u\|_j^2 = (u, u)_j^\gamma + (u, u)_0^\gamma, \quad (13)$$

then we obtain a Hilbert space $\overset{\circ}{H}_j^\gamma$ with norm equivalent to the norm of $\widetilde{W}_{x,y,2,\gamma}^{m,m}$. Using the apparatus of the mixed Fourier–Bessel transform, we prove the following assertion:

Theorem 1. *There exist positive constants c_1 and c_2 such that*

$$\widetilde{I}_\gamma(u, u) \geq C_1 \|u\|_{W_{2,\gamma}^m}^2 - C_2 \|u\|_{L_{2,\gamma}}^2. \quad (14)$$

Let us now consider the ordinary Dirichlet form of the type

$$\sum_{|\alpha|+k=m} \sum_{|\beta|+l=m} a_{\alpha\beta}^{kl}(z) D^{\alpha+k} u(z) D^{\beta+l} u(z) \quad (15)$$

with all the preceding restrictions on the coefficients $a_{\alpha\beta}^{kl}$, and, in addition, assume that they satisfy the inequality

$$|a_{\alpha\beta}^{kl}(z)| \leq C y^{k+l-2} \quad (k, l \geq 1). \quad (16)$$

For this form we construct the Dirichlet integral

$$\tilde{I}_\gamma(u, u) = \int_{\Omega^+} \sum a_{\alpha\beta}^{kl}(z) D^{k+\alpha} u(z) D^{\beta+l} u(z) y^{2\gamma} dz. \quad (17)$$

Then the following assertion holds.

Theorem 2. *There exist positive constants C'_1 and C'_2 such that*

$$\tilde{I}_\gamma(u, u) \geq C'_1 \|u\|_{W_{2,\gamma}^m}^2 - C'_2 \|u\|_{L_{2,\gamma}}^2. \quad (18)$$

Upper estimates in inequalities (14) and (18) are obtained quite simply. Let us now consider the following problem. Let $\mathcal{L}$ be a formally self-adjoint operator of order $2m$,

$$\mathcal{L} = (-1)^m \sum_{|\alpha|+k=|\beta|+l \leq m} D^{\alpha+k} (y^{2\gamma} a_{\alpha\beta}^{kl}(z) D^{\beta+l} u(z)) \quad (19)$$

with coefficients satisfying the restrictions indicated above. The non-self-adjoint case can be reduced to this one.

Dirichlet problem. In the domain Ω^+ a function $g \in H_m^\gamma(\Omega^+)$ is given, and an infinitely differentiable function h such that $\|h\|_{L_{2,\gamma}}$. Find a solution u of the equation

$$\mathcal{L}u = h$$

such that $g - u \in \dot{H}_m^\gamma$. The Dirichlet problem is solvable for arbitrary g and h , if the homogeneous problem has only the zero solution. The pair of equations $Qu = h$ and $Q^*v = g$, where Q^* is the operator formally adjoint to Q , forms a Fredholm pair. One can study the existence and properties of solutions of the equation

$$Qu - \lambda u = 0,$$

belonging to H_m^γ , etc.

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CITED LITERATURE

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