

On the Convergence of Solutions of the Cartwright-Littlewood Equation

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Abstract

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MATHEMATICS

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On the Convergence of Solutions of the Cartwright-Littlewood Equation

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Consider the equation

$$\ddot{y} + kf(y)\dot{y} + g(y) = kbp(t), \quad (1)$$

where the functions $f, g, p \in C^2(-\infty, +\infty)$, and k is a large parameter. Let the following assumptions be satisfied.

A. $p(t)$ is 2π -periodic, its mean value is equal to 0; $p(t + \pi) = -p(t)$; $p(t) = 0$ if and only if $t = \pi/2 \pmod{2\pi}$. Denote by $P(t)$ that antiderivative of $p(t)$ whose mean value is 0, and normalize $p(t)$ so that the greatest value of $P(t)$ is equal to 1 and corresponds to $t = \pi/2 \pmod{2\pi}$.

B. $f(y)$ is even, has a unique pair of zeros ± 1 , $f'(1) > 0$, and

$$\inf_{y \geq 2} f(y) > 0.$$

Put

$$F(y) = \int_0^y f(y) dy, \quad b_0 = F(-1).$$

By H and H_0 we denote the largest roots of the equations $F(y) = b$ and $F(y) = b_0$, respectively.

C. $g(y)$ is odd and $0 < l_1 < g' < l_2$.

These conditions are satisfied by the equation

$$\ddot{y} + k(y^2 - 1)\dot{y} + y = kb \cos t.$$

In what follows we assume that $b > b_0$, and k is sufficiently large. Everywhere by L we mean a positive constant independent of k and not fixed in the absence of an index, even within the same proposition. $\xi > 0$ always denotes a quantity $O(\exp\{-Lk\})$.

The case $0 < b < b_0$ was considered in detail in (1). There the hypothesis was also put forward that, in the case $b > b_0$, all solutions of equation (1) converge as $t \rightarrow +\infty$. Below the validity of this hypothesis is proved.

Theorem. If $b > b_0$, then there exists k_0 such that for all $k > k_0$ and any two solutions $y_1(t)$ and $y_2(t)$ of equation (1),

$$\lim_{t \rightarrow +\infty} |y_1(t) - y_2(t)| = 0, \quad \lim_{t \rightarrow +\infty} |\dot{y}_1(t) - \dot{y}_2(t)| = 0.$$

Below a scheme of the proof is given, with some details made more concrete.

Wendell proved (2) that for any $\varepsilon > 0$ one can specify $k_0(\varepsilon)$ and $n_0(\varepsilon)$ such that, for all $k > k_0(\varepsilon)$ and $n > n_0(\varepsilon)$, the inequalities hold (n are natural numbers)

$$|y(\pm\pi/2 + 2n\pi) \mp H| < \varepsilon, \quad |\dot{y}(\mp\pi/2 + 2n\pi)| < Lk^{1/2}. \quad (2)$$

In what follows all solutions are considered only for

$$t > \pi/2 + 2n_0(\varepsilon)\pi,$$

where ε is sufficiently small. It also follows from the arguments in (2) that there exists $\delta > 0$ such that $|y(t)| > H_0$ for $t \in (Z - 3\delta, Z + 3\delta)$ and $t \in (Z' - 3\delta, Z' + 3\delta)$. Here and below $Z = \pi/2 \pmod{2\pi}$, $Z' = -\pi/2 \pmod{2\pi}$, and in all statements formulated further only neighboring Z and Z' are considered.

Using the fact that $p(t) > 0$ for $t \in (Z', Z)$ and $p(t) < 0$ for $t \in (Z, Z')$, it is easy to obtain the following result.

Lemma 1. For every $d > 0$ one can specify an $A > 0$ such that $\dot{y}(t) > A$ for $t \in (Z' + d, Z - d)$ and $\dot{y}(t) < -A$ for $t \in (Z + d, Z' - d)$.

Lemma 1 makes it possible to determine the moments $U, K' \in (Z, Z')$ and $U', K \in (Z', Z)$ by means of the equalities $y(U) = 1$, $y(U') = -1$, $y(K) = \frac{1}{2}(1 + H_0)$, $y(K') = -\frac{1}{2}(1 + H_0)$. Let us also put $\dot{y}(u) = v$, $\dot{y}(U') = v'$. In what follows all results are formulated for one of the semiperiods (Z, Z') or (Z', Z) , but they are valid (with obvious modifications) also for the other.

Using (1), it is not difficult to obtain the estimates

$$Lk^{1/3} < v' < Lk^{1/2}; \quad (3)$$

$$\begin{aligned} v' + k[b_0 - F(y)] &\leq \dot{y}(t) \leq \\ &\leq v' + k[b_0 - F(y)] + L[(v'^2 + Lk(y+1))^{1/2} - v'] \end{aligned} \quad (4)$$

for $t \in (U', K)$, and moreover $\text{mes}(U', K) < Lk^{-1/2}$.

Now we can prove the following lemma.

Lemma 2. If $(t_1, t_2) \subset (U', K)$, then

$$L_1 \frac{\dot{y}(t_2)}{\dot{y}(t_1)} < \exp \left\{ - \int_{t_1}^{t_2} f(y) dt \right\} < L_2 \frac{\dot{y}(t_2)}{\dot{y}(t_1)}.$$

Proof. Put $\eta = y + 1$. Then from (4) we have

$$v' + kG(\eta) \leq \dot{y} \leq v' + kG(\eta) + \Phi(\eta); \quad (5)$$

$$\Phi(\eta) = L[(v'^2 + Lk\eta)^{1/2} - v'], \quad L\eta^2 < G(\eta) = b_0 - F(y) < L\eta^2. \quad (6)$$

Let $\eta_1 = \eta(t_1)$, $\eta_2 = \eta(t_2)$ and $\eta^* = k^{-1/3}$. Then

$$-k \int_{t_1}^{t_2} f(y) dt = \int_{\eta_1}^{\eta_2} \frac{k d\sigma}{\dot{y}}. \quad (7)$$

Consider the case $\eta_1 \leq \eta^* \leq \eta_2$. The estimates below follow from (3), (5), (6). From (5) we have

$$v' \leq \dot{y}(t_1) < Lv'. \quad (8)$$

Further, by virtue of (7),

$$-k \int_{t_1}^{t_2} f(y) dt \leq \int_0^{\eta^*} \frac{k dG}{v'} + \int_{\eta^*}^{\eta_2} \frac{k dG}{v' + kG},$$

whence

$$-k \int_{t_1}^{t_2} f(y) dt \leq \frac{kG(\eta^*)}{v'} + \ln \frac{v' + kG(\eta_2)}{v' + kG(\eta^*)} < \ln \frac{L\dot{y}(t_2)}{v'}. \quad (9)$$

Put

$$I = \int_{\eta^*}^{\eta_2} \frac{d\Phi}{v' + kG + \Phi}.$$

By virtue of (7),

$$-k \int_{t_1}^{t_2} f(y) dt + I > \int_{\eta^*}^{\eta_2} \frac{d(v' + kG + \Phi)}{v' + kG + \Phi}.$$

A direct calculation shows that $I < L$. Then

$$-k \int_{t_1}^{t_2} f(y) dt > \ln \frac{v' + kG(\eta_2) + \Phi(\eta_2)}{v' + kG(\eta^*) + \Phi(\eta^*)} - L > \ln \frac{L\dot{y}(t_2)}{v'},$$

Hence, and from (9),

$$\frac{L\dot{y}(t_2)}{v'} < \exp \left\{ - \int_{t_1}^{t_2} f(y) dt \right\} < \frac{L\dot{y}(t_2)}{v'},$$

and the assertion of the lemma follows from (8). Finally, the validity of the lemma when $\eta_1, \eta_2 \leq \eta^*$ and $\eta_1, \eta_2 \geq \eta^*$ follows from the case considered.

Let us now consider two arbitrary solutions $y_1(t)$ and $y_2(t)$ of equation (1). If X is a function of the solutions, then put $\Delta X = X(y_2) - X(y_1)$. Further, let

$$w = y_2 - y_1, \quad u = \Delta F/w, \quad \gamma = \Delta g/w \quad (l_1 < \gamma < l_2),$$

$$T = k \int_{Z'}^t u dt.$$

By the indices 1 and 2 we denote quantities connected with $y_1(t)$ or $y_2(t)$. Finally, put $c(t) = \dot{w} - \dot{T}w$. From (1) we obtain

$$c(t_2) - c(t_1) = - \int_{t_1}^{t_2} \gamma w dt. \quad (10)$$

The following two results are basic for what follows.

Lemma 3. There exists ζ^* with the following property: if $\zeta^* < w(Z) < k^{-11}$, then $0 < w(t) < k^{-5}$ for $t \in (Z, U'_2)$, and the ratio of any two of the quantities $kw(Z), kw(Z'), c(Z), c(Z'), c(U_1), c(U'_2)$ lies between two L' s.

Lemma 4. For any ζ' , if $|w(Z)| < \zeta'$, then $|w(t)|, |\dot{w}(t)|, |c(t)| < \zeta(\zeta')$ for $t \in (Z, U')$, where $U' = \min\{U'_1, U'_2\}$.

Lemmas 3 and 4 are analogues of Lemmas 24 and 25 of ⁽¹⁾, with the difference that in ⁽¹⁾ these lemmas are valid only for solutions belonging to a certain class. For solutions of this class, a lemma (Lemma 12 in ⁽¹⁾) analogous to Lemma 2 is valid, but its proof is based on facts which do not hold in the case under consideration. Using Lemma 2, and also Lemma 1 and the inequality $\text{mes}(U', K) < Lk^{-1/2}$, we may, in proving Lemmas 3 and 4, follow ⁽¹⁾, the arguments even being simplified; we omit the proofs and refer the reader directly to ⁽¹⁾ (Lemmas 13-25).

Lemma 5. If $w(Z) > \zeta^*$, then $w(t) > Lk^{-1}c(U_1)$ for $t \in (Z' + \delta, Z' + 2\delta)$.

Proof. Up to the first intersection after Z' of the trajectories $y_1(t)$ and $y_2(t)$ in the yt -plane, $w(t) > 0$, and, consequently, the inequality ⁽¹⁾, p. 55, is valid:

$$w(t) \int_{Z'}^t e^T dt \geq c(Z')\varphi(t) \left[\int_{Z'}^t e^T dt - \frac{w(Z')}{c(Z')} \psi(t) - \chi(t) \right], \quad (11)$$

where

$$\begin{aligned} \varphi(t) &= e^{-T} \int_{Z'}^t e^T dt, & \psi(t) &= l_2 \iint_{Z' \leq \xi \leq \eta \leq T} \exp\{T(\eta) - T(\xi)\} d\xi d\eta, \\ \chi(t) &= l_2 \iiint_{Z' \leq \xi \leq \eta \leq \zeta \leq t} \exp\{T(\xi) - T(\eta) + T(\zeta)\} d\xi d\eta d\zeta. \end{aligned}$$

Put now

$$\tau(t) = k \int_{Z'}^t f(y_1) dt.$$

Since $f(y_1) > L$ for $t \in (Z', Z' + 2\delta)$, it follows that

$$\tau \geq Lk(t - Z').$$

Therefore, for $t \in (Z' + \delta, Z' + 2\delta)$,

$$\int_{Z'}^t e^{-\tau} dt < \int_{Z'}^t e^{-Lk(t-Z')} dt < Lk^{-1}; \quad (12)$$

$$Lk^{-1} < e^{-\tau} \int_{Z'}^t e^{\tau} dt < Lk^{-1}. \quad (13)$$

$$\iint_{Z' < \xi < \eta < t} \exp\{\tau(\xi) - \tau(\eta)\} d\xi d\eta = \int_{Z'}^t d\eta \left(e^{-\tau(\eta)} \int_{Z'}^{\eta} e^{\tau(\xi)} d\xi \right) < Lk^{-1}. \quad (14)$$

By Lemma 3, $0 < w(t) < k^{-5}$ for $t \in (Z' + \delta, Z' + 2\delta)$, and therefore $\dot{\tau} = T + O(k^{-4})$ and $\exp\{\pm T\} = \exp\{\pm \tau\}(1 + O(k^{-4}))$. Hence it follows that the estimates (12)–(14) remain valid if τ is replaced in them by T . Then

$$\psi < L \left(\int_{Z'}^t e^{T(\eta)} d\eta \right) \left(\int_{Z'}^t e^{-T(\xi)} d\xi \right) < Lk^{-1} \int_{Z'}^t e^T dt,$$

$$\chi < L \left(\int_{Z'}^t e^{T(\zeta)} d\zeta \right) \left(\iint_{Z' < \xi < \eta < t} \exp\{T(\xi) - T(\eta)\} d\xi d\eta \right) < Lk^{-1} \int_{Z'}^t e^T dt$$

by virtue of (12) and (14). Hence, from Lemma 3 and from (11) and (13), we obtain, for $t \in (Z' + \delta, Z' + 2\delta)$,

$$w(t) > Lk^{-1}c(Z') > Lk^{-1}c(U_1).$$

Lemma 6. If $|w(Z)| < k^{-11}$, then

$$|c(U'_2)| < (1 - Lk^{-1})|c(U_1)| + \zeta.$$

Proof. If $|w(Z)| \leq \xi^*$, then the assertion of the lemma is trivial in view of Lemma 4. Let now $w(Z) > \xi^*$. Then $w(t) > 0$ for $Z \leq t \leq U'_2$, and by Lemma 3 and from (10) we have

$$c(U'_2) - c(U_1) < -L \int_{U_1}^{U'_2} w dt.$$

Hence, by Lemma 5,

$$c(U'_2) - c(U_1) < -Lk^{-1}c(U_1),$$

which proves the lemma.

Proof of the theorem. By Lemma 6, for any two solutions of equation (1) for which $|w(Z)| < k^{-11}$, either $c(U_1), c(U'_2) = O(\zeta)$, or $c(U_1), c(U'_2)$ decrease in absolute value (though slowly for large k) and eventually become quantities of order ζ . Then, by Lemma 4, $w(t), \dot{w}(t) = O(\zeta)$ for all sufficiently large t . This proves convergence to within ζ . The proof of true convergence is the same as in (1) (§§ 55–56). As a result we obtain convergence under the condition $|w(Z)| < k^{-11}$. But from (2) it follows that then $\lim_{t \rightarrow +\infty} w(t) = 0$ for any two solutions of equation (1), and $\lim_{t \rightarrow +\infty} \dot{w}(t) = 0$, since $\dot{w}(t)$ is bounded as $t \rightarrow +\infty$.

The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.

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