

# ON THE CONSTRUCTION OF CUBATURE FORMULAS WITH THE SMALLEST NUMBER OF NODES

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**Abstract**

**Full Text**

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**MATHEMATICS**

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# ON THE CONSTRUCTION OF CUBATURE FORMULAS WITH THE SMALLEST NUMBER OF NODES

*(Presented by Academician V. I. Smirnov, 17 IV 1967)*

Let  $\Omega$  be a domain in  $n$ -dimensional Euclidean space and let  $p(x)$  be a weight function, nonnegative for  $x \in \Omega$ , such that the integrals

$$\mu_{\alpha_1 \dots \alpha_n} = \int_{\Omega} p(x) x_1^{\alpha_1} \dots x_n^{\alpha_n} dx$$

exist, the moments of the domain  $\Omega$  and of the weight function  $p(x)$ ,  $\mu_{0 \dots 0} > 0$ .

Consider the problem of constructing a cubature formula

$$\int_{\Omega} p(x) f(x) dx \cong \sum_{j=1}^N C_j f(x^{(j)}), \quad (1)$$

exact for all polynomials of degree not higher than  $m$ , where  $m = 2l$  is even. The smallest possible number of nodes for which formula (1) can exist is equal to

$$M(n, l) = (n + l)! / n! l!.$$

In what follows we assume that in (1)  $N = M(n, l)$ .

Renumber the monomials

$$x_1^{\alpha_1} \dots x_n^{\alpha_n}, \quad \alpha_1 \geq 0, \dots, \alpha_n \geq 0,$$

and introduce for them the notation  $\{\varphi_i(x)\}_{i=1}^{\infty}$ . The numbering is carried out so that monomials of lower degree receive a smaller number, while monomials of the same degree are numbered in any order. In particular,  $\varphi_1(x) = 1$ .

Write that formula (1) is exact for all polynomials of degree  $\leq m$ , in other words, exact when  $f = \varphi_i(x)$ ,

$$\sum_{j=1}^N C_j \varphi_i(x^{(j)}) = \int_{\Omega} p(x) \varphi_i(x) dx, \quad i = 1, 2, \dots, M(m, n). \quad (2)$$

We have obtained a system of  $M(m, n)$  equations in  $(n+1)M(n, l)$  unknowns.

Introduce three square matrices of order  $N$ : the matrix  $X$ , determined by the nodes of the cubature formula (1),

$$X = \begin{pmatrix} \varphi_1(x^{(1)}), & \varphi_2(x^{(1)}), \dots, \varphi_N(x^{(1)}) \\ \varphi_1(x^{(2)}), & \varphi_2(x^{(2)}), \dots, \varphi_N(x^{(2)}) \\ \vdots & \vdots \\ \varphi_1(x^{(N)}), & \varphi_2(x^{(N)}), \dots, \varphi_N(x^{(N)}) \end{pmatrix};$$

the diagonal matrix  $C$ , determined by the coefficients of formula (1),

$$C = [C_1, \dots, C_N]$$

and the Gram matrix of the system  $\{\varphi_i(x)\}_{i=1}^N$

$$G = ((\varphi_i, \varphi_k))_{i,k=1}^N. \quad (3)$$

Here it is denoted

$$(\varphi_i, \varphi_k) = \int_{\Omega} p(x) \varphi_i(x) \varphi_k(x) dx.$$

It is known that the system (2) can be written in the form of a single matrix equation [1]

$$X'CX = G. \quad (4)$$

The prime denotes the transposition operation. It follows from equality (4) that the number of nodes  $N$  in (1) is the smallest.

The matrix (3) is positive definite, and for it the representation [2]

$$G = \Gamma' \Gamma, \quad (5)$$

holds, where  $\Gamma$  is an upper triangular matrix. The nonuniqueness of the representation (5) reduces to the fact that one may change the sign of all elements of any row of the matrix  $\Gamma$ .

Introduce the notation

$$\Gamma^{-1} = \begin{pmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1N} \\ 0 & \beta_{22} & \dots & \beta_{2N} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \beta_{NN} \end{pmatrix}.$$

In transforming equation (4), the following assertion is used. The polynomials

$$F_i(x) = \beta_{1i}\varphi_1(x) + \beta_{2i}\varphi_2(x) + \dots + \beta_{ii}\varphi_i(x), \quad i = 1, 2, \dots, N,$$

form an orthonormal system. In view of what was said above about the nonuniqueness of representation (5), the polynomial  $F_i(x)$  is determined by the matrix  $G$  up to sign.

Let us give the proof. Denote by  $\beta^{(i)}$  the vector coinciding with the  $i$ -th column of the matrix  $\Gamma^{-1}$ . We have

$$\begin{aligned} (F_i, F_j) &= \left( \sum_{k=1}^i \beta_{ki}\varphi_k, \sum_{l=1}^j \beta_{lj}\varphi_l \right) = \sum_{k=1}^i \sum_{l=1}^j \beta_{ki}\beta_{lj}(\varphi_k, \varphi_l) = (G\beta^{(i)}, \beta^{(j)}) = \\ &= (\Gamma'\Gamma\beta^{(i)}, \beta^{(j)}) = (\Gamma\beta^{(i)}, \Gamma\beta^{(j)}) = (e^{(i)}, e^{(j)}) = \delta_i^{(j)}. \end{aligned}$$

Here  $e^{(i)}$  is the  $i$ -th column of the identity matrix, and  $\delta_i^{(j)}$  is the Kronecker symbol. The assertion is also true in the case when  $\varphi_1, \dots, \varphi_N$  are linearly independent elements of a Hilbert space.

Equation (4) can be brought to the form

$$FF' = C^{-1}, \quad (6)$$

where

$$F = X\Gamma^{-1} = \begin{pmatrix} F_1(x^{(1)}), & F_2(x^{(1)}), & \dots, & F_N(x^{(1)}) \\ F_1(x^{(2)}), & F_2(x^{(2)}), & \dots, & F_N(x^{(2)}) \\ \dots & \dots & \dots & \dots \\ F_1(x^{(N)}), & F_2(x^{(N)}), & \dots, & F_N(x^{(N)}) \end{pmatrix}. \quad (7)$$

Let us note that equation (6) could have been obtained by writing down the fact that the cubature formula (1) is exact for  $f = F_i F_k$ ,  $i, k = 1, 2, \dots, N$ .

From (6) it is seen that

$$C_i^{-1} = F_1^2(x^{(i)}) + F_2^2(x^{(i)}) + \dots + F_N^2(x^{(i)}), \quad i = 1, 2, \dots, N. \quad (8)$$

In particular, the coefficients  $C_i$  are positive.

The rows of the matrix (7), by virtue of (6), are orthogonal. If  $a$  is one of the nodes of formula (1), then its remaining nodes lie on the algebraic hypersurface

$$F_1(a)F_1(x) + F_2(a)F_2(x) + \dots + F_N(a)F_N(x) = 0. \quad (9)$$

This circumstance can be used in constructing formula (1). For brevity, we shall say that the point  $a$  determines the hypersurface (9).

Let us indicate examples of the construction of the cubature formula (1). For  $m = 2$  we obtain a new algorithm for constructing a cubature formula exact for polynomials of the second degree. From the system of monomials  $1, x_1, \dots, x_n$  we construct an orthonormal system  $F_i(x)$ ,  $i = 1, \dots, n + 1$ . We choose the point  $x^{(1)}$  so that among the numbers  $F_i(x^{(1)})$ ,  $i = 2, 3, \dots, n + 1$ , there are some different from zero. Denote by  $L_1$  the hyperplane determined by the point  $x^{(1)}$ . On  $L_1$  we choose a point  $x^{(2)}$  so that the hyperplane  $L_2$  determined by  $x^{(2)}$  is not parallel to  $L_1$ . At the intersection of  $L_1$  and  $L_2$  we take a point  $x^{(3)}$  so that the hyperplane  $L_3$  determined by it is not parallel to any of the hyperplanes  $L_1$  and  $L_2$ . At the intersection of  $L_1, L_2, L_3$  we choose a point  $x^{(4)}$ , and so on. As a result we obtain  $n + 1$  points  $x^{(1)}, \dots, x^{(n+1)}$ —the nodes of the cubature formula. Another method of constructing formula (1) for  $m = 2$  is considered in (3).

Now let  $n = 2$ ,  $m = 4$ . We assume that the domain  $\Omega$  is symmetric with respect to both coordinate axes and that the weight has the same symmetry property:  $p(x, y) = p(-x, y) = p(x, -y)$  for  $(x, y) \in \Omega$ . By virtue of the assumptions on  $\Omega$  and  $p(x, y)$ , the moment  $\mu_{ik} = 0$  if at least one of the numbers  $i$  and  $k$  is odd.

We shall suppose that the origin  $(x_1, y_1) = (0, 0)$  is a node of the desired cubature formula. The curve determined by the point  $(0, 0)$  has equation

$$\alpha x^2 + \gamma y^2 = \delta, \quad (10)$$

where  $\alpha = \mu_{04}\mu_{20} - \mu_{22}\mu_{02}$ ,  $\gamma = \mu_{40}\mu_{02} - \mu_{22}\mu_{20}$ ,  $\delta = \mu_{40}\mu_{04} - \mu_{22}^2$ ; here  $\delta > 0$  and at least one of the numbers  $\alpha, \gamma$  is positive. If, for example,  $\alpha > 0$ , then as one more node we take  $(x_2, y_2) = (\sqrt{\delta/\alpha}, 0)$ . The curve determined by this node,

$$\mu_{04}x^2 - \mu_{22}y^2 + \frac{\alpha}{\mu_{20}}\sqrt{\frac{\delta}{\alpha}}x = 0$$

intersects the curve (10) in four real points  $(x_i, y_i)$ ,  $i = 3, 4, 5, 6$ . The points  $(x_i, y_i)$ ,  $i = 1, \dots, 6$ , may be taken as the nodes of a cubature formula exact for polynomials of the fourth degree.

For  $n = 1$  we obtain a quadrature formula exact for polynomials of degree  $m = 2l$  and having  $N = l + 1$  nodes. The formula depends on the parameter  $a$ —

the abscissa of one of the nodes. The remaining nodes are determined from the algebraic equation (9) of degree  $N - 1$ . Let  $I$  be the smallest interval containing  $\Omega$ . If  $a$  lies outside  $I$  or coincides with one of its endpoints, then the roots of equation (9) are real, distinct, and lie inside  $I$ . If  $a$  is one of the roots of the orthogonal polynomial  $F_{N+1}(x)$ , then the roots of (9) coincide with the remaining roots of  $F_{N+1}(x)$ . In this case formula (8) is well known <sup>(4)</sup>.

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*Note: Figure translations are in progress. See original paper for figures.*

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