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1968

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Abstract

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UDC 517.946

MATHEMATICS

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THE GREEN MATRIX OF A HOMOGENEOUS PARABOLIC BOUNDARY-VALUE PROBLEM FOR SYSTEMS WITH DISCONTINUOUS COEFFICIENTS

(Presented by Academician I. N. Vekua on 20 II 1968)

Parabolic boundary-value problems for systems whose coefficients may have discontinuities on smooth hypersurfaces not meeting the boundary of the domain in which the solution of the problem is sought have been studied in sufficient detail ^(1,2).

Here we shall carry out a detailed study of the kernel of the integral operator solving the homogeneous problem—the Green matrix. In doing so we shall constantly use the authors' investigations on estimates of the Green matrix for a problem with smooth coefficients ⁽³⁾ and the following consideration: in form a problem with discontinuities (a transmission problem) is more general than an ordinary parabolic boundary-value problem, but, as will be shown below, essentially for the entire subsequent discussion, the transmission problem in the whole space for systems with constant coefficients containing only the group of terms principal in the parabolic sense is a special case of the ordinary boundary-value problem for a half-space (if systems are considered!).*

1. Formulation of the problem. Let Ω be a domain of the space E_n with boundary S , divided by an $(n-1)$ -dimensional surface S_1 into two subdomains** Ω_1 and Ω_2 , where S_1 and S have no common points, and

$$Q_1 = [0, T] \times \Omega_1, \quad Q_2 = [0, T] \times \Omega_2, \quad Q = [0, T] \times \Omega$$

are cylindrical domains of the space E_{n+1} . Denote by Γ_1 and Γ the lateral surfaces of the cylinders Q_1 and Q . The domain Ω and the surfaces S and S_1 may be either finite or infinite.

Consider the problem of finding a solution of the parabolic, in the sense of Petrovskii, system

$$Lu = \begin{cases} L_1(t, x, D_t, D_x)u_1(t, x) = f_1(t, x), & (t, x) \in Q_1, \\ L_2(t, x, D_t, D_x)u_2(t, x) = f_2(t, x), & (t, x) \in Q_2, \end{cases} \quad (1)$$

$$L_m(t, x, D_t, D_x) = D_t - \sum_{|k| \leq 2b} A_k^{(m)}(t, x) D_x^k \quad (m = 1, 2),$$

in the domain $Q \setminus \Gamma_1$, satisfying on the surface Γ the boundary condition

$$B(t, x, D_x)u_2|_{\Gamma} = 0, \quad (2)$$

on the surface Γ_1 the transmission condition

$$[C_1(t, x, D_x)u_1 - C_2(t, x, D_x)u_2]|_{\Gamma_1} = 0, \quad (3)$$

and for $t = 0$ the initial condition

$$u_m|_{t=0} = \varphi_m(x) \quad (m = 1, 2). \quad (4)$$

* This circumstance was noted and substantially used, in a somewhat different situation, in an interesting work ⁽⁴⁾.

** All results are also valid in the case of division into any finite number of subdomains.

Here $A_k^{(m)}(t, x)$ are matrices of size $N \times N$; B is a matrix of size $bN \times N$, whose elements $B_{ij}(t, x, D_x)$ are linear differential operators of order $r_i \leq 2b - 1$, while C_1, C_2 are matrices of size $2bN \times N$, whose elements $C_{ij}^{(1)}(t, x, D_x), C_{ij}^{(2)}(t, x, D_x)$ are similar operators of order $m_i \leq 2b - 1$. The operators B and L_2 are connected by the algebraic Lopatinskii condition ⁽⁵⁻⁷⁾, and (C_1, C_2) and (L_1, L_2) by the algebraic condition of joint covering ^(1,2).

N. V. Zhitarashu established in ⁽²⁾ that the problem described above is correctly solvable in Hölder spaces and in certain Sobolev-Slobodetskii spaces $W_p^{(l)}$.

In the present paper it is established that the solution of problem (1)–(4) is given by the formula

$$u_m(t, x) = \sum_{j=1}^2 \left(\int_{\Omega_j} G_m(t, x; 0, \xi) \varphi_j(\xi) d\xi + \int_0^t d\tau \int_{\Omega_j} G_m(t, x; \tau, \xi) f_j(\tau, \xi) d\xi \right), \quad (t, x) \in Q_m, \quad (5)$$

and a detailed study is carried out of the Green matrix $G = (G_1, G_2)$ of problem (1)–(4).

2. The conjugation problem and the general boundary-value problem in a half-space. Consider the problem, fundamental for everything that follows, of finding a vector-function $u(t, x) = (u_1(t, x), u_2(t, x))$ satisfying the parabolic system

$$L_{m0}(D_t, D_x)u_m \equiv \left(D_t - \sum_{|k|=2b} A_k^{(m)} D_x^k \right) u_m(t, x) = f_m(t, x),$$

$$x \in E^{(m)} \quad (m = 1, 2), \quad (6)$$

the conjugation condition

$$[C_{10}(D_x)u_1 - C_{20}(D_x)u_2]_{x_n=0} = g(t, x'), \quad (7)$$

the initial condition

$$u_m|_{t=0} = 0, \quad x \in E^{(m)} \quad (m = 1, 2), \quad (8)$$

and such that $u_m(t, x) \in L_2([0, T] \times E^{(m)})$, where $E^{(1)} = E_n^+$, $E^{(2)} = E_n^-$.

Instead of the vector-function $u_2(t, x)$, introduce the vector-function $v(t, x)$, defining it by the equality $v(t, x) = u_2(t, x', -x_n)$; then problem (6)–(8) passes into the boundary-value problem of finding the vector-function

$$w(t, x) = \begin{pmatrix} u_1(t, x) \\ v(t, x) \end{pmatrix}, \quad w(t, x) \in L_2([0, T] \times E_n^+),$$

satisfying the conditions:

$$L_0(D_t, D_x)w = f(t, x), \quad (9)$$

$$C_0(D_x)w|_{x_n=0} = g(t, x'), \quad (10)$$

$$w|_{t=0} = 0, \quad (11)$$

where

$$L_0(D_t, D_x) = \begin{pmatrix} L_{10}(D_t, D_x) & 0 \\ 0 & L_{20}(D_t, D_{x'}, -D_{x_n}) \end{pmatrix},$$

$$C_0(D_x) = (C_{10}(D_x), -C_{20}(D_{x'}, -D_{x_n})), \quad f(t, x) = \begin{pmatrix} f_1(t, x) \\ f_2(t, x', -x_n) \end{pmatrix}.$$

It is easily verified that the condition of joint covering appearing in the conjugation problem (6)–(8) coincides with the Lopatinskii condition for solvability (for arbitrary g) of problem (9)–(11).

Thus, everything necessary for a detailed investigation of problem (6)–(8)–the Poisson kernels and the Green matrix–can be obtained from the preceding studies^(6–8) of boundary value problems for parabolic systems in a half-space.

3. Structure and estimates of the Green matrix of problem (1)–(4).
The following fundamental result is valid.

Theorem 1. If the coefficients $A_k^{(m)}$ of the system (1_m) belong to the class* $C_{x,t}^{l+\alpha,(l+\alpha)/2b}(Q_m)$, the coefficients of the operators B_{ij} and $C_{ij}^{(m)}$ belong respectively to the classes $C_{x,t}^{2b-r_i+l+\alpha,(2b-r_i+l+\alpha)/2b}(\Gamma)$ and $C_{x,t}^{2b-m_i+l+\alpha,(2b-m_i+l+\alpha)/2b}(\Gamma_1)$, and the surfaces S, S_1 are of class $C^{2b+l+\alpha}$, $0 < \alpha < 1$, $l \geq 0$ is an integer, then there exists a matrix

$$G(t, x; \tau, \xi) \equiv (G_1(t, x; \tau, \xi), G_2(t, x; \tau, \xi)),$$

where $G_m(t, x; \tau, \xi)$ is defined in $\overline{Q_m} \times \overline{Q}$ for $t > \tau$, has $2b$ derivatives with respect to x and one derivative with respect to t , and possesses the following properties:

- a) $G_m(t, x; \tau, \xi) = Z_m(t, \tau, x, \xi) - v_m(t, x; \tau, \xi)$, where Z_m is the fundamental matrix of solutions (f.m.s.) of the system (1_m) ; for $t > \tau$, G_m satisfies in t, x the system (1_m) , G_1 and G_2 satisfy the conjugation condition (3), and G_2 satisfies the boundary condition (2); $v_m|_{t=\tau} = 0$ when at least one of the points x or ξ does not lie on S, S_1 .
- b) If $\varphi_m \in C^\alpha(\Omega_m)$, $f_m \in C_{x,t}^{\alpha,\alpha/2b}(Q_m)$, then the vector-function (5) is a solution of problem (1)–(4).
- c) For the derivatives of the matrix G_m the estimates

$$|G_m^{(k_0,k)}(t, x; \tau, \xi)| \leq C(t - \tau)^{-(n+2bk_0+|k|)/2b} \Psi_c(t - \tau, x - \xi) \quad (12)$$

$$(2bk_0 + |k| \leq 2b + l);$$

$$|\Delta_x G_m^{(k_0,k)}(t, x; \tau, \xi)| \leq C|x - x_0|^\alpha (t - \tau)^{-(n+2b+l+\alpha)/2b} \Psi_c(t - \tau, y - \xi), \quad (13)$$

$$(2bk_0 + |k| = 2b + l);$$

$$|\Delta_t G_m^{(k_0,k)}(t, x; \tau, \xi)| \leq C(t - t_0)^{(2b+l-2bk_0-|k|+\alpha)/2b} (t_0 - \tau)^{-(n+2b+l+\alpha)/2b} \times \Psi_c(t - \tau, x - \xi) \quad (14)$$

$$(l < 2bk_0 + |k| \leq 2b + l, \tau < t_0 < t),$$

hold, where

$$G_m^{(k_0,k)} = D_t^{k_0} D_x^k G_m, \quad \Psi_c(t - \tau, x - \xi) = \exp \left\{ -c \frac{|x - \xi|^{2b/(2b-1)}}{(t - \tau)^{1/(2b-1)}} \right\},$$

$$\Delta_x f(t, x) = f(t, x) - f(t, x_0), \quad \Delta_t f(t, x) = f(t, x) - f(t_0, x),$$

y is the point among x and x_0 closest to the point ξ . The same estimates are valid for the derivatives of v_m , except that everywhere, instead of $|x - \xi|$ and $|y - \xi|$, one must put respectively $|x - \xi| + \rho(\xi)$ and $|y - \xi| + \rho(\xi)$, where $\rho(\xi)$ is the least of the distances from the point ξ to S and S_1 . The constants C, c depend on the norms of the coefficients of the system, of the boundary operator and of the conjugation operators, on various characteristics of Ω_m and S, S_1 , on the numbers $2b, n, \alpha, T$, and on the constants in the inequalities connected with the parabolicity, the Lopatinskii condition, and the compatible covering.

- d) If the coefficients of the system, of the boundary operator, and of the conjugation operator do not depend on t , then for $G_m \equiv G_m(t - \tau, x, \xi)$ the inequalities (12)–(14) are valid, with the additional factor $\exp\{A(t - \tau)\}$ (A is some nonnegative number) appearing on their right-hand sides, and the constants in them do not depend on T .

4. Consequences. I. As in the case of the problem with smooth coefficients ⁽³⁾, from Theorem 1 one can obtain results on the Green matrix of a homogeneous elliptic boundary value problem with discontinuous coefficients generated by a parabolic one. Consider the elliptic problem

* For the definition of all classes of functions and surfaces occurring here, see ⁽⁶⁾.

$$\sum_{|k| \leq 2b} A_k^{(m)}(x) D_x^k u_m + \lambda u_m = f_m(x) \quad (x \in \Omega_m, \quad m = 1, 2); \quad (15_m)$$

$$B(x, D_x) u_2|_S = 0; \quad (16)$$

$$[C_1(x, D_x) u_1 - C_2(x, D_x) u_2]|_{S_1} = 0, \quad (17)$$

$\operatorname{Re} \lambda > A$ (A is the number discussed in item d) of Theorem 1), which is generated by the parabolic problem (1)–(4).

Theorem 2. If the coefficients of the system (15_m) , of the operators B_{ij} and $C_{ij}^{(m)}$ belong respectively to $C^{l+\alpha}(\overline{\Omega}_m)$, $C^{2b-r_i+l+\alpha}(S)$, and $C^{2b-m_i+l+\alpha}(S_1)$, and $S, S_1 \in C^{2b+l+\alpha}$, $l \geq 0$, then there exists a matrix $\Phi(x, \xi; \lambda) = (\Phi_1(x, \xi; \lambda), \Phi_2(x, \xi; \lambda))$, where $\Phi_m(x, \xi; \lambda)$ is defined in $\overline{\Omega}_m \times \overline{\Omega}$ for $x \neq \xi$, has $2b + l$ derivatives with respect to x , and possesses the following properties:

a)

$$\Phi_m(x, \xi; \lambda) = \int_0^\infty G_m(t, x, \xi; \lambda) dt = \varphi_m(x, \xi; \lambda) - w_m(x, \xi; \lambda),$$

where $(G_1(t, x, \xi; \lambda), G_2(t, x, \xi; \lambda))$ is the Green matrix of the parabolic problem generating problem (15)–(17); $\varphi_m(x, \xi; \lambda)$ is the f.s. of the system (15_m); for $x \neq \xi$, Φ_m satisfies the homogeneous system (15_m), Φ_1 and Φ_2 satisfy the conjugation condition (17), and Φ_2 satisfies the boundary condition (16).

b) If $f_m \in G^\alpha(\bar{\Omega}_m)$, then the vector-valued function $u(x) = (u_1(x), u_2(x))$,

$$u_m(x) = \sum_{j=1}^2 \int_{\Omega_j} \Phi_m(x, \xi; \lambda) f_j(\xi) d\xi \quad (m = 1, 2)$$

is a solution of problem (15)–(17).

c) For the matrix Φ_m the estimates

$$|D_x^k \Phi_m(x, \xi; \lambda)| \leq C e^{-\delta|x-\xi|} \begin{cases} 1, & n + |k| < 2b, \\ 1 + |\ln|x-\xi||, & n + |k| = 2b, \\ |x-\xi|^{-n-|k|+2b}, & n + |k| > 2b, \end{cases} \quad (|k| \leq 2b + l) \quad (18)$$

$$|\Delta_x^{x_0} D_x^s \Phi_m(x, \xi; \lambda)| \leq C |x - x_0|^\alpha e^{-\delta|y-\xi|} |y - \xi|^{-n-|s|+2b-\alpha} \quad (|s| = 2b + l, \delta = c_0(\operatorname{Re} \lambda - A)^{1-2b}). \quad (19)$$

For the matrix w_m , the estimates (18), (19) hold in which $|x - \xi|$ and $|y - \xi|$ are replaced respectively by $|x - \xi| + \rho(\xi)$ and $|y - \xi| + \rho(\xi)$.

II. On the basis of the results given above one can also obtain exact estimates for the Green functions of fractional negative powers of the operator corresponding to problem (15)–(17), as was done in (3) for the case of smooth coefficients.

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Received
13 II 1968

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