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Abstract

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MATHEMATICS

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METRIC THEORY OF LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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In the present paper a theory of linear systems of differential equations is constructed that is adequate to the metric theory of dynamical systems. As is well known, the behavior of the trajectories of a dynamical system Δ , defined by a smooth vector field on an n -dimensional manifold of class C^2 (which we shall assume to be compact), in a neighborhood of a given trajectory $x_0(t)$ is described by the system in variations

$$d\delta x/dt = D_x f(x_0(t))\delta x. \quad (1)$$

If the trajectory $x_0(t)$ is fixed, then we are dealing with the fixed linear system

$$\dot{x} = A(t)x. \quad (2)$$

(The results obtained for such systems up to 1965 are presented in ^(1,3).)

However, from the point of view of the metric theory of dynamical systems (see ⁽²⁾, Ch. VI), what is essential is not the behavior of an individual trajectory of the system Δ , but of sets of trajectories whose set of initial points has positive invariant measure. By the Krylov–Bogolyubov theorem (see ⁽²⁾, p. 514, Theorem 24), there exists a normalized invariant measure on the system Δ . Accordingly, we shall be interested not in individual systems (2), but in certain collections of them. Suppose now that the manifold is embedded in Euclidean space and that the vector field is defined in its neighborhood. It is easy to see that if $x_k(t)$ are trajectories of the system Δ , then $\tilde{x}(t) = \lim_{k \rightarrow \infty} x_k(t_k + t)$ (the limit being uniform on intervals) is also a trajectory of the system Δ , and that if the system in variations along the trajectory $x_k(t)$ is $d\delta x/dt = A_k(t)\delta x$, then the system in variations along the trajectory $\tilde{x}(t)$ is $d\delta x/dt = \tilde{A}(t)\delta x$, where $\tilde{A}(t) = \lim_{k \rightarrow \infty} A_k(t_k + t)$ (the limit being uniform on intervals). Thus we obtain a natural continuous mapping of the dynamical system Δ into the dynamical system D of shifts of matrix functions $A(t)$ (this system is described on pp. 533–

535 in ⁽²⁾, admittedly for numerical functions, but the difference is immaterial). It will be more convenient for us, having fixed an arbitrary trajectory $x_0(t)$ of the system Δ , to consider the subsystem $\Delta_{x_0(t)}$ of the system Δ , defined on the closure of the trajectory $x_0(t)$. The first basic step consists in detaching ourselves from this dynamical system and considering the dynamical system D_A of shifts of the matrix function $A(t) \equiv D_x f(x_0(t))$ (a subsystem of the system D), which is a continuous image of the system $\Delta_{x_0(t)}$. (We note that, for example, the strict ergodicity of the dynamical system Δ implies the strict ergodicity of the dynamical system D_A , etc.)

Thus, let a matrix function $A(t)$ be given, bounded and uniformly continuous on the line. We shall study system (2). (System (2) may already fail to be a system in variations for any dynamical system Δ ; therefore this problem is more general than the preceding one.)

One of the basic tools in studying system (2) is its reduction to triangular form

$$\dot{u} = P(t)u; \quad P(t) = \begin{pmatrix} p_{11}(t), \dots, p_{1n}(t) & & \\ & \ddots & \\ 0 & & p_{nn}(t) \end{pmatrix} \quad (3)$$

by a Perron transformation $x = U(t)u$ (see ⁽¹⁾, pp. 261-272). Let us fix such a transformation and consider the dynamical system D_P of shifts of the matrix function $P(t)$.

At first our aim is to study the connections between the dynamical systems D_A and D_P (we shall denote their spaces by R_A and R_P , respectively). The following simple reasoning explains why this is needed. The functions $\varphi_i(\tilde{P}) \equiv \tilde{p}_{ii}(0)$, where $\tilde{P}(t) \in R_P$, are continuous on R_P . Therefore, by Birkhoff's ergodic theorem (see ⁽²⁾, pp. 480-490), for almost all $\tilde{P} \in R_P$ (in the sense of any invariant measure on D_P) there exist the means

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \varphi_i(\tilde{P}(\tau)) d\tau = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \tilde{p}_{ii}(\tau) d\tau,$$

and hence the system $\dot{u} = \tilde{P}(t)u$ is regular (see ⁽¹⁾, p. 141, Lyapunov's theorem). The question is whether in D_A almost all (in the sense of any invariant measure on D_A) $\tilde{A}(t) \in R_A$ are such that the system $\dot{x} = \tilde{A}(t)x$ is regular. (From Theorem 2 there follows, in particular, a positive answer to this question.)

Define a mapping F of the system D_A onto the system D_P as follows:

$$F(\tilde{A}(t) = \lim_{k \rightarrow \infty} A(t_k + t)) = \tilde{P}(t) = \lim_{k \rightarrow \infty} P(t_k + t)$$

(the sign $\lim$ here denotes not a limit, but any of the limit points of the sequence, since this mapping is multivalued in both directions).

The fundamental role of the mapping F is based on the following proposition, whose proof see in ⁽⁶⁾.

Lemma 1. *If $F(\tilde{A}) = \tilde{P}$, then there exists a Perron transformation $x = \tilde{U}(t)u$ reducing the system $\dot{x} = \tilde{A}(t)x$ to the triangular form $\dot{u} = \tilde{P}(t)u$.*

Using the fact that F and F^{-1} carry closed sets into closed sets, one can estimate invariant measures of sets in D_A in terms of invariant measures of their images in D_P , and conversely; this makes it possible to prove the following lemma. (In what follows, an invariant measure is always assumed to be a normalized regular Carathéodory-Lebesgue measure; see ⁽²⁾, p. 461, axiom V.)

Lemma 2. *For every invariant measure μ on D_A , for every set $M \subseteq R_A$ such that $\mu(M) > 0$ (respectively $= 1$), there exists an invariant measure ν on D_P such that $\nu(F(M)) > 0$ (respectively $= 1$).*

Conversely, for every invariant measure ν on D_P and every set $N \subseteq R_P$ such that $\nu(N) > 0$ (respectively $= 1$), there exists an invariant measure μ on D_A such that $\mu(F^{-1}(N)) > 0$ (respectively $= 1$).

Definition 1 (see ⁽⁶⁾). We shall call λ a **probable exponent** of system (2) if some Perron transformation $x = U(t)u$ reduces system (2) to a triangular form (3) such that, for some i , on the dynamical system D_P there is an invariant measure ν such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \tilde{p}_{ii}(\tau) d\tau = \lambda$$

for almost every (in the sense of the measure ν) $\tilde{P}(t) \in R_P$.

It follows from Lemma 2 that Definition 1 is equivalent to the following definition.

Definition 2. We shall call λ a **probable exponent** of system (2) if, on the dynamical system D_A , there exists an invariant measure μ such that for almost all (in the sense of the measure μ) $\tilde{A}(t) \in R_A$, the system $\dot{x} = \tilde{A}(t)x$ has λ as one of its characteristic exponents.

We shall denote the set of probable exponents of system (2) by Λ_p , or $\Lambda_p(A)$, and call it the probable spectrum of system (2).

Theorem 1. For every generalized solution of system (2)

$$\tilde{x}(t) = \lim_{k \rightarrow \infty} x_k(t_k + t)$$

(i.e., an ordinary solution, a shift of an ordinary solution or a limiting solution, see ⁽⁴⁾), the numbers

$$\bar{\lambda} = \overline{\lim}_{t-\tau \rightarrow +\infty} \frac{1}{t-\tau} \ln \frac{\|\tilde{x}(t)\|}{\|\tilde{x}(\tau)\|}, \quad \underline{\lambda} = \underline{\lim}_{t-\tau \rightarrow +\infty} \frac{1}{t-\tau} \ln \frac{\|\tilde{x}(t)\|}{\|\tilde{x}(\tau)\|}$$

belong to $\Lambda_p(A)$.

The special exponents Ω^0 and ω^0 of system (2) (see, for example, ⁽¹⁾, p. 191) also belong to $\Lambda_p(A)$. The theorem follows from ⁽⁶⁾ and Lemma 2.

Theorem 2. In the sense of any invariant measure on the dynamical system D_A , almost all $\tilde{A}(t) \in R_A$ are such that the system $\dot{x} = \tilde{A}(t)x$ is statistically regular (see ⁽⁵⁾).

The theorem follows from ⁽⁶⁾ and Lemma 2.

The following is also true (see ⁽⁶⁾, Theorem 4).

Theorem 3. If system (2) is statistically regular, then its greatest characteristic exponent is right-proper, and the least is left-proper (for an explanation of these terms, see ⁽¹⁾, p. 162, Definition 13.1.1).

Definition 3. We shall call system (2) **biregular** if there exists a Perron transformation $x = U(t)u$ reducing it to triangular form (3), such that the limits

$$\lambda_i = \lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t p_{ii}(\tau) d\tau \quad (i = 1, 2, \dots, n).$$

exist.

Theorem 4. In the sense of any transitive invariant measure on D_A , almost all $\tilde{A}(t) \in R_A$ are such that the system $\dot{x} = \tilde{A}(t)x$ is biregular, and its characteristic exponents are the same for almost all (in the sense of this measure) $\tilde{A}(t) \in R_A$.

The purpose of introducing biregular systems (obviously, this is a subclass of regular systems) is that for them, along with Theorem 4, the following holds.

Theorem 5. Suppose system (2), by Perron transformations $x = U(t)u$ and $x = V(t)v$, is reduced to triangular form respectively

$$\dot{u} = P(t)u, \quad \dot{v} = Q(t)v,$$

and moreover

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t p_{ii}(\tau) d\tau = \lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t q_{ii}(\tau) d\tau = \lambda_i \quad (4)$$

$$(i = 1, 2, \dots, n).$$

Suppose the λ_i are all distinct. Then $P(t) \equiv Q(t)$.

From Theorems 4 and 5 it follows:

Theorem 6. Suppose μ is a transitive invariant measure on D_A , and suppose that almost all (in the sense of the measure μ) $\tilde{A}(t) \in R_A$ are such that the system $\dot{x} = A(t)x$ has no multiple characteristic exponents. Then the mapping

F , defined above, is finite-valued almost everywhere (in the sense of the measure μ).

Let us now consider an important special case of system (2), namely, suppose that the dynamical system D_A is strictly ergodic. (This will be the case, for example, if the matrix $A(t)$ is almost periodic in t .)

From Lemma 2 and Theorem 4 it follows that

Theorem 7. *Let the dynamical system D_A be strictly ergodic. Then for almost every $\tilde{A}(t) \in R_A$ (in the sense of the unique invariant measure that exists on D_A), every $\tilde{P}(t) = F(\tilde{A}(t))$ (the mapping F may be multivalued!) has the same (but only, perhaps, numbered in a different order!)*

$$\lambda_i = \lim_{t \rightarrow \pm\infty} \frac{1}{t} \int_0^t p_{ii}(\tau) d\tau \quad (i = 1, 2, \dots, n),$$

and the set of these λ_i coincides with the probable spectrum $\Lambda_p(A)$.

Corollary 1. *If the dynamical system D_A is strictly ergodic, then the cardinality of the probable spectrum $\Lambda_p(A)$ of system (2) does not exceed n (the order of system (2)).*

Corollary 2. *If the dynamical system D_A is strictly ergodic, and $\Lambda_p(A)$ consists of n distinct numbers, then the mapping F is finite-valued almost everywhere on D_A (in the sense of the invariant measure on D_A) (the number of values is $\leq n!$).*

Corollary 3*. *If the dynamical system D_A is strictly ergodic, then for almost every $\tilde{A}(t) \in R_A$ (in the sense of the invariant measure on D_A) the largest characteristic exponent of the system $\dot{x} = \tilde{A}(t)x$ is equal to Ω^0 , and the smallest is equal to ω^0 .*

Proofreader's note. We have learned that in Oseledets' article ⁽⁷⁾ there are assertions close to Lemma 2 and Theorem 4.

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* Follows from Theorems 1 and 7.

Note: Figure translations are in progress. See original paper for figures.

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